Exam #1 Solutions

Problem 1.

(1) Let us compute the components of the vector $\vec{u} = \overrightarrow{IA}$:

$$\vec{u} = (x_A - x_I, y_A - y_I, z_A - z_I)$$
$$\vec{u} = (-3, 2, 1)$$

Let us compute the components of the vector $\vec{v} = \overrightarrow{IB}$:

$$\vec{v} = (x_B - x_I, y_B - y_I, z_B - z_I)$$
$$\vec{v} = (3, -2, 3)$$

(2) In general, a parametric equation for a line L through a point $M_0(x_0, y_0, z_0)$ and directed by a non-null vector $\vec{u} = (u_1, u_2, u_3)$ is:

$$L: \begin{cases} x(t) &= x_0 + tu_1 \\ y(t) &= y_0 + tu_2 \\ z(t) &= z_0 + tu_3 \end{cases}$$

In our situation, we find the following parametric equations:

$$L_1: \begin{cases} x(t) &= 1 - 3t \\ y(t) &= -1 + 2t \\ z(t) &= -6 + t \end{cases}$$

$$L_2: \begin{cases} x(t) &= 1 + 3t \\ y(t) &= -1 - 2t \\ z(t) &= -6 + 3t \end{cases}$$

(3) We start by finding a vector $\vec{w}$ which is orthogonal to both $\vec{u}$ and $\vec{v}$ by computing the cross-product $\vec{w} = \vec{u} \times \vec{v}$:

$$\begin{aligned} \vec{w} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3 & 2 & 1 \\ 3 & -2 & 3 \end{vmatrix} \\ &= \begin{vmatrix} 2 & 1 \\ -2 & 3 \end{vmatrix} \vec{i} - \begin{vmatrix} -3 & 1 \\ 3 & 3 \end{vmatrix} \vec{j} + \begin{vmatrix} -3 & 2 \\ 3 & -2 \end{vmatrix} \vec{k} \\ &= 8\vec{i} + 12\vec{j} \\ &= (8, 12, 0) \end{aligned}$$

The line L_3 goes through I and is directed by $\vec{w}$. Therefore a parametric equation is given by:

$$L_3: \begin{cases} x(t) &= 1 + 8t \\ y(t) &= -1 + 12t \\ z(t) &= -6 \end{cases}$$

(4) In order to determine whether point $C(-1, 2, -6)$ belong to the line L_3, we try to solve for t the following system of equations:

$$\begin{cases} -1 &= 1 + 8t \\ 2 &= -1 + 12t \\ -6 &= -6 \end{cases} .$$

We quickly see that $t = -1/4$ is the solution of the first equation but not the second equation, therefore there is no solution. Conclusion: the line L_3 does not go through C.

(5) We check whether $\overrightarrow{IC} = (-2, 3, 0)$ is orthogonal to $\overrightarrow{IA} = (-3, 2, 1)$ by computing their dot product:

$$\overrightarrow{IC} \cdot \overrightarrow{IA} = (-2) \times (-3) + 3 \times 2 + 0 \times 1$$
$$\overrightarrow{IC} \cdot \overrightarrow{IA} = 12 .$$

Conclusion: $\overrightarrow{IC} \cdot \overrightarrow{IA} \neq 0$ therefore $\overrightarrow{IC}$ is not orthogonal to $\overrightarrow{IA}$.
We check whether $\overrightarrow{IC} = (-2, 3, 0)$ is orthogonal to $\overrightarrow{IB} = (3, -2, 3)$ by computing their dot product:

$$\overrightarrow{IC} \cdot \overrightarrow{IB} = (-2) \times 3 + 3 \times (-2) + 0 \times 3$$
$$\overrightarrow{IC} \cdot \overrightarrow{IB} = -12 .$$

Conclusion: $\overrightarrow{IC} \cdot \overrightarrow{IB} \neq 0$ therefore $\overrightarrow{IC}$ is not orthogonal to $\overrightarrow{IB}$.

Problem 2.

(1) By definition, the velocity $\vec{v}(t)$ is given by $\vec{v}(t) = \vec{r}'(t)$, so here we find:

$$\vec{v}(t) = (\cos(t), \cos(t), -\sqrt{2}\sin(t)) .$$

By definition, the speed $v(t)$ is given by $v(t) = \|v(t)\|$, so here we find:

$$\begin{aligned} v(t) &= \sqrt{\big(\cos(t)\big)^2 + \big(\cos(t)\big)^2 + \big(-\sqrt{2}\sin(t)\big)^2} \\ &= \sqrt{2\big(\cos(t)\big)^2 + 2\big(\cos(t)\big)^2} \\ &= \sqrt{2} . \end{aligned}$$

Note that this motion has constant speed (but not constant velocity).

By definition, the acceleration $\vec{a}(t)$ is given by $\vec{a}(t) = \vec{v}'(t)$, so here we find:

$$\vec{a}(t) = (-\sin(t), -\sin(t), -\sqrt{2}\cos(t)) .$$

(2) The parametrization $f(t)$ is not a parametrization by arclength, since $v(t) \neq 1$. The arclength parameter s is given by the formula:

$$s = \int_0^t v(\tau)\, d\tau .$$

In this situation the formula yields:

$$\begin{aligned} s &= \int_0^t \sqrt{2}\, d\tau \\ s &= \sqrt{2}t \end{aligned}$$

We find an arclength parametrization by rewriting $f(t)$ in terms of s, given that $t = s/\sqrt{2}$:

$$\begin{aligned} f(t) &= f(s/\sqrt{2}) \\ &= \Big(\sin(s/\sqrt{2}), \sin(s/\sqrt{2}), \sqrt{2}\cos(s/\sqrt{2})\Big) . \end{aligned}$$

and we get the following arclength parametrization:

$$g(s) = \Big(\sin(s/\sqrt{2}), \sin(s/\sqrt{2}), \sqrt{2}\cos(s/\sqrt{2})\Big) .$$

(3) By definition, the unit tangent vector $\vec{T}(t)$ is given by $\vec{T}(t) = \frac{\vec{v}(t)}{v(t)}$, so here we find:

$$\vec{T}(t) = \Big(\cos(t)/\sqrt{2}, \cos(t)/\sqrt{2}, -\sin(t)\Big) .$$

By definition, the principal unit normal vector $\vec{N}(t)$ is given by $\vec{N}(t) = \frac{\vec{T}'(t)}{||\vec{T}'(t)||}$. In this situation we find $\vec{T}'(t) = \left(-\sin(t)/\sqrt{2}, -\sin(t)/\sqrt{2}, -\cos(t)\right)$ and $\left\|\vec{T}'(t)\right\| = 1$, so that:

$$\vec{N}(t) = \left(-\sin(t)/\sqrt{2}, -\sin(t)/\sqrt{2}, -\cos(t)\right) .$$

By definition, the unit binormal vector $\vec{B}(t)$ is given by $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$, so here we find:

$$\begin{aligned}
\vec{B}(t) &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{cos(t)}{\sqrt{2}} & \frac{cos(t)}{\sqrt{2}} & -\sin(t) \\ -\frac{sin(t)}{\sqrt{2}} & -\frac{sin(t)}{\sqrt{2}} & -\cos(t) \end{vmatrix} \\
&= \begin{vmatrix} \frac{cos(t)}{\sqrt{2}} & -\sin(t) \\ -\frac{sin(t)}{\sqrt{2}} & -\cos(t) \end{vmatrix} \vec{i} - \begin{vmatrix} \frac{cos(t)}{\sqrt{2}} & -\sin(t) \\ -\frac{sin(t)}{\sqrt{2}} & -\cos(t) \end{vmatrix} \vec{j} + \begin{vmatrix} \frac{cos(t)}{\sqrt{2}} & \frac{cos(t)}{\sqrt{2}} \\ -\frac{sin(t)}{\sqrt{2}} & -\frac{sin(t)}{\sqrt{2}} \end{vmatrix} \vec{k} \\
&= \left(-\frac{\cos(t)^2}{\sqrt{2}} - \frac{\sin(t)^2}{\sqrt{2}}\right)\vec{i} + \left(\frac{\cos(t)^2}{\sqrt{2}} + \frac{\sin(t)^2}{\sqrt{2}}\right)\vec{j} + 0\vec{k} \\
&= -\frac{1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{2}}\vec{j} \\
&= \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)
\end{aligned}$$

NB: We can observe that the unit binormal vector $\vec{B}(t)$ is constant. This is saying that the curve is contained in a plane having $\vec{B}$ as a normal vector.

(4) As recalled in the question, the curvature $\kappa(t)$ is given by $\kappa(t) = \frac{||\vec{T}'(t)||}{v(t)}$. In our situation, we have already computed $||\vec{T}'(t)|| = 1$ and $v(t) = \sqrt{2}$, thus the answer is:

$$\kappa(t) = \frac{1}{\sqrt{2}}$$

(5) The radius of curvature is equal to the inverse of the curvature:

$$\begin{aligned} R(t) &= \frac{1}{\kappa(t)} \\ &= \sqrt{2} . \end{aligned}$$

We notice that the radius of curvature is constant: $R(t) = R = \sqrt{2}$. Thus it is reasonable to conjecture that the curve is a circle of radius $R = \sqrt{2}$. However, note that there are many other curves of constant curvature, for example a circular helix.

(6) By definition, the length of the curve between $t = a$ and $t = b$ is given by $L = \int_a^b v(t)\,dt$. In our situation:

$$
\begin{aligned}
L &= \int_0^{2\pi} v(t)\,dt \\
&= \int_0^{2\pi} \sqrt{2}\,dt \\
&= 2\pi\sqrt{2}\,.
\end{aligned}
$$

Note that this curve is periodic with period equal to 2π, since clearly $f(2\pi) = f(0)$. Thus the length of the curve between $t = 0$ and $t = 2\pi$ is the total length of the curve. The total length we found, $L = 2\pi\sqrt{2}$, is consistent with our previous conjecture that the curve is a circle of radius $R = \sqrt{2}$: the circumference of such a circle is indeed $2\pi R = 2\pi\sqrt{2}$.

(7) The Cartesian equation of a sphere is given by:

$$x^2 + y^2 + z^2 = R^2$$

where R is the radius of the sphere.
Here the coordinates $(x(t), y(t), z(t))$ of the moving point verify:

$$
\begin{aligned}
x(t)^2 + y(t)^2 + z(t)^2 &= \big(\sin(t)\big)^2 + \big(\sin(t)\big)^2 + \big(\sqrt{2}\cos(t)\big)^2 \\
x(t)^2 + y(t)^2 + z(t)^2 &= 2\big(\cos(t)\big)^2 + 2\big(\sin(t)\big)^2 \\
x(t)^2 + y(t)^2 + z(t)^2 &= 2\,.
\end{aligned}
$$

This shows that the path lies on the sphere centered at the origin with radius $R = \sqrt{2}$.

(8) In order to show that the path lies on the plane with Cartesian equation $x - y = 0$, we need to check that the coordinates $(x(t), y(t), z(t))$ of the moving point verify this equation. It is straightforward:

$$
\begin{aligned}
x(t) - y(t) &= \sin(t) - \sin(t) \\
x(t) - y(t) &= 0\,.
\end{aligned}
$$

This shows that the path lies on the plane with Cartesian equation $x - y = 0$.

(9) The previous two questions show that the curve is the intersection of the sphere S centered at the origin with radius $R = \sqrt{2}$ and the plane P with equation $x - y = 0$. In general, the intersection of a plane and a sphere is a circle (when it is not empty). Here note that the plane P goes through the origin and the sphere S is centered at the origin, thus their intersection is a circle of radius $\sqrt{2}$ centered at the origin.

Problem 3.

Let us denote by $M(t)$ the point with coordinates $\big(x(t), y(t), z(t)\big)$ in 3-dimensional space, recording the position of the falling object at time t. As usual, we also denote $\vec{r}(t) = \overrightarrow{OM(t)}$ the position vector, $\vec{v}(t) = \vec{r}'(t)$ the velocity and $\vec{a}(t) = \vec{r}''(t)$ the acceleration.

Let us assume that the objet's initial position is $(0, 0, H)$. Thus the following initial conditions are satisfied:

$$\begin{cases} \vec{r}(0) = (0, 0, H) \\ \vec{v}(0) = (0, 0, 0) . \end{cases}$$

By assumption, the only force ifluencing the object's motion is the gravitational force $\vec{F} = m\vec{g}$, where m is the mass of the falling object and $\vec{g}$ is the gravitational field given by $\vec{g} = (0, 0, -g)$. Thus the gravitational force is:

$$\vec{F} = (0, 0, -mg) .$$

Newton's second law of motion states that:

$$m\vec{a} = \vec{F} .$$

In our case, this gives the following expression for the acceleration:

$$\vec{a}(t) = (0, 0, -g) .$$

We now integrate in order to find the velocity:

$$\begin{aligned} \vec{v}(t) &= \int_0^t \vec{a}(u)\, du + \vec{v}(0) \\ &= \int_0^t (0, 0, -g)\, du + (0, 0, 0) \\ &= (0, 0, -gt) . \end{aligned}$$

And we integrate again to find the position vector:

$$\begin{aligned} \vec{r}(t) &= \int_0^t \vec{v}(u)\, du + \vec{r}(0) \\ &= \int_0^t (0, 0, -gt)\, du + (0, 0, H) \\ &= (0, 0, H - \frac{1}{2}gt^2) . \end{aligned}$$

The object hits the ground at the time t_{max} such that $z(t_{\text{max}}) = 0$, which yields:

$$H - \frac{1}{2}g{t_{\text{max}}}^2 = 0$$

$$t_{\text{max}} = \sqrt{\frac{2H}{g}} .$$

When this happens, the velocity of the object is:

$$\begin{aligned}\vec{v}(t_{\text{max}}) &= \vec{v}\left(\sqrt{\frac{2H}{g}}\right) \\ &= (0, 0, -g\sqrt{\frac{2H}{g}}) \\ &= (0, 0, -\sqrt{2gH})\end{aligned}$$

and the speed of the object is:

$$\begin{aligned}v(t_{\text{max}}) &= \|\vec{v}(t_{\text{max}})\| \\ &= \sqrt{2gH}\ .\end{aligned}$$

Note that, as it turns out, the answer is independent of the mass of the object.