Reading group **"Higher Teichmüller-Thurston spaces"**
following Olivier Guichard's habilitation thesis

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Higgs bundles and Hitchin components

LECTURE 1/4.

References (just for starters)

- Hitchin's original paper (selfduality equations ... 19
- Background notes for AIM workshop 2008
- Bradlow: video archive of GEAR retreat 2012
- Goldman: Higgs bundles and geometric ... 2008
- Guichard: Autrans notes 2012 (HdR thesis of course

I Introduction

Blablabla.

Disclaimer: I know nothing. + (These notes lack details and proofs)

Situation:

Γ discrete $\xrightarrow{\text{rep}}$ G Lie group

Higher Teichmüller theories

- Hitchin theory — G real split semi simple
- Maximal reps — G Hermitian type
- Positive reps. — G ...
- Anosov reps (?)

Setting: $\Gamma = \pi_1(M) = \pi$ M (manifold) $M = S$ closed oriented surface, $\chi(S) < 0$ (soon S, X Riemann surface)

G (real ~~split semisimple~~) $G = GL(n, \mathbb{R})$ (or $SL(n, \mathbb{R})$ soon)   // $GL(n, \mathbb{C})$, $SL(n, \mathbb{C})$ at first

most of the time.

Goal: $\rho : \pi \to G$ $\xleftrightarrow[\rho \mapsto E_\rho]{\text{monodromy}}$ G - local system ($G = GL_n\mathbb{C}$ for the moment)

$\Updownarrow$ (G-principal flat bundle)

Explain this: holonomy; $E_\rho(V)$

(E, D): E smooth (complex) vector bundle; D flat connection $F_D = 0$

Corlette, Donaldson; Hitchin, Simpson

$\Updownarrow$ Hitchin equations, metrics

Higgs bundle (E, φ): E holo G-vector bundle; $\varphi \in \Omega^{1,0}(\mathrm{Ad}\, E)$

holomorphic (algebraic) objects

+ statement in terms of moduli spaces.

Properties of ~~Hitchin~~ Higgs moduli space,

Hitchin representations

Outline

I. Introduction
II. Bundles
III. Connections
IV. Holo. bundles
V. Hermitian bundles
VI Higgs bundles
VII Stability
VIII Isomorphisms between moduli
IX Higgs moduli space
X Reality
XI Hitchin fibration
XII Hitchin component
XIII SL(2) example

II Fibre bundles

(Ref: Steenrod, topology of fiber bundles; ....)

what we need to know about them, why we need them...

A fibre bundle consists at least of:

E "total space"

$\downarrow p$ "projection" (surjective)

M "base"

F: "type fiber"

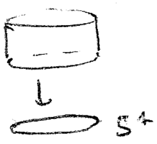
such that $\forall x \in M$ $E_x := p^{-1}(x) \approx F$

condition: locally trivial (locally a product):

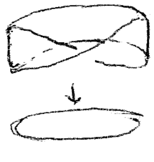
$\forall x \in M\ \exists \mathcal{U} \ni x \quad p^{-1}(\mathcal{U}) \overset{\varphi_{\mathcal{U}}}{\approx} \mathcal{U} \times F.$

Examples

1) cylinder (product)



2) Möbius band



(not trivial)

3) coverings (discrete fiber)

4) Group quotients

5) Vector bundles, in part usual tensor bundles TM, T^*M, $\Lambda^k T^*M$, $\bigotimes_{i=1}^{n} TM$ etc

def A fiber bundle is the data of:

$$\begin{array}{ccc} E & F & \text{Covering } \{U_i\} \\ \downarrow p & & \text{transition functions} \quad g_{ij} : U_i \cap U_j \to G \\ M & & \end{array}$$

+ cocycle condition $g_{ik} = g_{ij} g_{jk}$

idea: $\varphi_i : p^{-1}(U_i) \to U_i$

$\varphi_j \circ \varphi_i^{-1} : U_{ij} \times F \to U_{ij} \times$

$(x, y) \mapsto (x, g_{ij}(x$

only data needed

$G \subset \mathrm{Aut}(F)$

or more generally $G \curvearrowright F$ is called the <u>structure group</u>

<u>In brief</u>: equivalence class of atlas

morphism of fiber bundle

subbundle

reduction / loosening of the structure group. e.g. Riemannian metric $\leftrightarrow$ reduction to $O(n)$

automorphisms ((when vector bundle) <u>gauge transformations</u>)

examples of structure groups.

<u>Note</u>: for us, everything is smooth.

<u>defs</u>:
- A (real) <u>vector bundle</u> is a bundle where $F = V$ real v. space and $G \subset GL(V)$
- A (smooth) <u>complex vector bundle</u> is a vector bundle where $F = V$ complex vector space and $G \subset GL_{\mathbb{C}} V$.
- A <u>holomorphic (structure on a complex) vector bundle</u> is a choice of g_{ij} such that g_{ij} is holomorphic ($\bar{\partial} g_{ij} = 0$).

  <u>Note</u> Here M is a complex manifold. (Holomorphic bundle: tighter equivalence relation on atlases
- A <u>flat (structure on a) bundle</u> is a choice of g_{ij} such that g_{ij} is (locally) constant ($d g_{ij} = 0$). (again, tighter equiv. rel on atlases).

  <u>Note</u> If M is a complex manifold, flat complex vector bundle $\Rightarrow$ holomorphic vector bund

  see later for more on flat bundles.

Examples.

<u>Associated bundles, principal bundles</u>

The data of F plays no role! Can be replaced by any space that G acts on:

$\to$ Associated bundles.

In particular, F can be replaced by G ($G \curvearrowright G$ by left multiplication)

$\to$ Associated <u>principal bundle</u> (fiber = structure group)

Associated bundles share properties... examples. like being flat

Bundles associated to a representation (homomorphism) $\rho : \pi = \pi_1(M) \longrightarrow \underset{\text{Lie group}}{G}$

Consider $\begin{matrix} \tilde{M} \\ \downarrow \\ M \end{matrix}$. This is a π-bundle where $\pi = \pi_1(M)$.

Given $\rho : \pi_1(M) \longrightarrow G$, we have an associated bundle $\underline{E_\rho(G)}$.

Considering $Ad \circ \rho : \pi_1(M) \longrightarrow Aut(\mathfrak{g})$, we also have an associated bundle $\underline{E_\rho(\mathfrak{g})}$.

If $G \subset GL(V)$ where V real/complex vector space, we also have an ass. bundle $\underline{E_\rho(V)}$.

(Note: if $G = GL(V)$, $E_\rho(\mathfrak{g}) =$ "$Ad\ E_\rho(V)$" $= End\ E_\rho(V)$).

Prop: these bundles are all flat, because $\begin{matrix} \tilde{M} \\ \downarrow \\ M \end{matrix}$ obviously is.

Vocabulary: $E_\rho(G)$ is called a G-local system, which can be defined as

- A flat principal bundle
- A locally constant sheaf
- A "twisted product" $M \times_\rho G$ where $M \times_\rho G = \tilde{M} \times G / \pi$ $\quad \gamma \cdot (\tilde{m}, g) = (\gamma . \tilde{m}, \rho(\gamma) g)$

Remark: In the case where ρ is the holonomy of a geometric structure on M,

- $E_\rho(G)$ can be thought of as the sheaf of germs of geometric functions on M
- $E_\rho(\mathfrak{g})$ ——————————————————— vector fields ————

Lemma (Homotopy $\Rightarrow$ isomorphism)

$\begin{matrix} E \\ \downarrow \\ M \times [0,1] \end{matrix}$ $\quad E_{|M \times \{0\}}$ and $E_{|M \times \{1\}}$ are isomorphic.

Cor $\forall \rho_0 : \pi \longrightarrow G$ $\quad \exists \underset{\rho_0}{\mathcal{U}} \subset Hom(\pi, G)$ $\quad \forall \rho \in \mathcal{U}$ $\quad E_\rho \cong E_{\rho'}$.

idea: Fix E, change flat structure on E.

II Connections

Refs : many available

let $\begin{array}{c} E \\ \downarrow \\ M \end{array}$ ((real) complex) vector bundle. (structure group $G \subset GL(V)$).

Notation : $\Gamma(E)$ = sections
$\Omega^k(E)$ = k-forms

How can we:
- define horizontal vectors ? → def of a connection with horizontal distribution
- compare tangent spaces ? → def by parallel transport
- differenciate sections ? → def as covariant derivative.

def A connection is a linear operator $\Omega^0(E) = \Gamma(E) \xrightarrow{D} \Omega^1(E)$
such that $\forall f \in \mathcal{C}^\infty(M)$ $\forall s \in \Gamma(E)$ $D(fs) = (df)\, s + f\, Ds$. (Leibniz rule)

One often writes $D_X s \in \Gamma(E)$ (instead of $Ds(X)$).
X: vector field on M; s: section

Prop The space of connections is an affine space modeled on $\Omega^1(\mathrm{End}\, E)$.

Locally (on $\mathcal{U}_\alpha$) $s(x) = \sum s_i(x)\, e_i(x)$

$$Ds(x) = \sum ds_i(x)\, e_i(x) + s_i(x)\, A_\alpha \cdot e_i(x)$$

« $D = d + A$ » with $A_\alpha \in \Omega^1(\mathcal{U}_\alpha, \mathrm{End}\, E)$ connection one-form.

examples
- Levi-Civita connection
- Chern connection (cf later)
- flat connection on a flat vector bundle

Note D induces connections on all bundles associated to E

parallel transport If $\gamma : x \longrightarrow y$ is a path in M, this is an operator
$T_\gamma : E_x \longrightarrow E_y$.

holonomy of a flat connection. If $\gamma : x \longrightarrow x$ is a loop $T_\gamma : E_x \longrightarrow E_x$
only depends on $[\gamma]$, this defines $hol_x : \pi_1(M, x) \longrightarrow GL(E_x)$.

G-connection : $T_\gamma \in G$, equivalently connection one-form $\in \Omega^1(\mathfrak{g})$

<u>Prop</u> If $\rho : \pi_1(M) \longrightarrow G$, then $\mathrm{hol}(D) = \rho$
where D is the flat connection on $E_\rho(V)$ $\qquad (G \subset GL(V))$.

<u>Curvature</u> $\Omega^0(E) \xrightarrow{D} \Omega^1(E) \xrightarrow{D} \Omega^2(E)$

$F_D := D \cdot D \in \Omega^2(\mathrm{End}\, E)$ $\qquad$ (or $\Omega^2(\mathfrak{g})$).

D flat $\iff F_D = 0$.

<u>Conclusion</u> : At this point we have explained this part :

$$\begin{tikzcd}
\rho : \pi = \pi_1(M) \longrightarrow G \arrow[rr, leftrightarrow, "\leftarrow\ \text{monodromy}", "\rho \mapsto E_\rho(G)\ \to"'] & & G\text{-local system on } M \arrow[d, leftrightarrow, "\text{associated}"] \\
& & (E, D) \text{ with } F_D = 0 \arrow[llu, "\text{holonomy}"', "\rho \mapsto E_\rho(V)"]
\end{tikzcd}$$

$\boxed{\text{IV Holomorphic vector bundles}}$ $\leftarrow$ start of <u>LECTURE 2/4</u> (recap first...) $\qquad$ (*) (foot note)

<u>Recall</u> • A holo. vector bundle is a G-vector bundle on a <u>complex</u> manifold M with type fiber $F = V$ complex vector space, $G \subset GL_{\mathbb{C}}(V)$, g_{ij} holomorphic

• A flat complex vector bundle on M is holomorphic

(on M, $d = \underset{\partial}{d^{1,0}} + \underset{\bar\partial}{d^{0,1}}$ $\qquad dg_{ij} = 0 \Rightarrow \bar\partial g_{ij} = 0$)

Alternatively, a holomorphic structure on a complex vector bundle $E \longrightarrow M$ complex manifold can be defined as the choice of an operator $\bar\partial_E : \Omega^0(E) \longrightarrow \Omega^{0,1}(E)$ such that

- $\forall f \in C^\infty(M)\ \forall s \in \Omega^0(M)$ $\bar\partial_E(fs) = (\bar\partial f)s + f(\bar\partial_E s)$ (Leibniz rule) $\qquad$ $\bar\partial$ operator on M
- $\bar\partial_E^{\,2} = 0$ (integrability, automatic if $\dim_{\mathbb{C}} M = 1$)

<u>lemma</u> $\bar\partial_E$ exists and is unique and determines holo structure)

*) Recall that a complex str on M induces a decomposition $T^{\mathbb{C}}M := T^{\mathbb{R}}M \otimes \mathbb{C} = T^{1,0}M \oplus T^{0,1}_{(M)}$
$\Omega^1(M) = \Omega^{1,0}_{(M)} + \Omega^{0,1}(M)$ $\quad \Omega^1(E) = \Omega^{1,0}(E) \oplus \Omega^{0,1}(E)$ $\quad d = \partial + \bar\partial$ etc.

(*) (continuation of footnote)

If D is a connection on E, $D = D^{1,0} + D^{0,1}$.

Note Flat $\Rightarrow$ holo is seen in terms of connections:

D flat $\Rightarrow$ $\bar\partial_E = D^{0,1}$ is a pseudo connection (a holo. structure).

Conclusion At this point, if M is a complex manifold, we have a map

$\rho : \pi_1(M) \to G \rightsquigarrow E$ holomorphic vector bundle on M

However, this arrow contains a loss of information (not invertible)

So we need to keep some additional data, holomorphic data if possible.

V Hermitian metrics on complex vector bundles

let $E \to M$ (smooth) complex vector bundle ($G = GL_n(\mathbb{C})$, say).

def A Hermitian metric h on E is a section $h \in \Gamma(E^* \otimes E^*)$ with Hermitian symmetr

i.e $\forall x \in M$ h_x (linear) Hermitian product on E_x ($h_x : \underset{\mathbb{C}\text{-linear}}{E_x} \times \underset{\mathbb{C}\text{-antilinear}}{E_x} \to \mathbb{C}$ + $h(v,u) = \overline{h(u,v)}$)

and h_x varies smoothly.

Note: • h defines Hermitian metrics on all bundles associated to E

• h defines adjunction on operators, e.g if $D : \Omega^0(E) \to \Omega^1(E)$ is a connection,

$D^* : \Omega^1(E) \to \Omega^0(E)$ (defined by $h(Ds, w) = h(s, D^*w)$)

Note: Choice of Hermitian metric $\Leftrightarrow$ Reduction of structure group to $U(n)$ $\Leftrightarrow$ Consistent decomposition $E(\mathfrak{g}) = E_H(\mathfrak{u}) \oplus E_H(\mathfrak{m})$

where $\mathfrak{g} = \underset{\mathrm{Lie}(U(n))}{\mathfrak{u}} \oplus \underset{\text{Hermitian matrices}}{\mathfrak{m}}$ Cartan decompositi

def: A connection D on (E, h) is called <u>unitary</u> if:

- $\forall s, s'$ $d(h(s,s')) = h(Ds, s') + h(s, Ds')$
- equivalently, $Dh = 0$
- equivalently, D is a $U(n)$ connection

Prop Let D be a connection on (E,h). Then D decomposes uniquely as

$$D = D_h + \Psi_h \quad \text{where} \quad \begin{cases} D_h \text{ unitary connection} \\ \Psi_h \in \Omega^1(E) \text{ Hermitian}: \ (\Psi \in \Omega^1(E_H(m))). \\ \Psi_h^* = \Psi_h \end{cases}$$

Prop Let $\bar\partial_E$ be a holo structure on E, h Hermitian metric.
Then $\exists!\ D_{\bar\partial_E, h}$ unitary connection such that $\bar\partial_E = D^{0,1}_{\bar\partial_E, h}$.

$D_{\bar\partial_E, h}$ is called the Chern connection of h

Note The Chern connection has no reason to be flat.

Energy of a metric, harmonic metrics

Here $M = X = S$ with a complex structure (Riemann surface).
And assume $E = (E, D)$ is a flat bundle.

def Let h be a metric on E, consider the decomposition $D = D_h + \Psi_h$.
The energy of h is defined as $\mathcal{E}(h) = \int_X \|\Psi_h\|^2_{g,h}\, d\mathrm{vol}_g$

Note: we have chosen here a metric g in the conformal class of X.

h is called harmonic if $\mathcal{E}(h)$ is minimal. Note: this does not depend on the choice of g.

Prop: h is harmonic if and only if $D_h^* \Psi_h = 0$.

VI Hitchin equations, Higgs bundles (finally!)

Let $E \longrightarrow M$ complex vector bundle
$M = X$
Riemann surface

The Hitchin equations are :
("self duality")

$$\boxed{\begin{cases} F_{D_h} + [\varphi, \varphi^*] = 0 \\ \bar\partial_E \varphi = 0 \end{cases}}$$

These are equations on a metric h, given either :

(1) A flat connection D on E. In this case $D = D_h + \Psi_h$
and $\varphi = \Psi_h^{(1,0)} \in \Omega^{1,0}(\mathrm{End}\, E)$ (and $\varphi^* = \Psi_h^{0,1}$)
and $\bar\partial_E = D_h^{(0,1)}$

(2) A holomorphic structure $\bar\partial_E$ and $\varphi \in \Omega^{1,0}(\mathrm{End}\, E)$. In this case D_h = Chern connection

Thm: D flat, h harmonic $\iff$ h satisfies the Hitchin equations

The precise meaning of the theorem is the following:
Given D flat, harmonic, let $D = D_h + \Psi_h$, $\bar\partial_E = D_h^{(0,1)}$ and $\varphi = \Psi_h^{1,0}$.
Then these satisfy the Hitchin equations.

Conversely, let $\bar\partial_E$ be a holomorphic structure, $\varphi \in \Omega^{1,0}(\mathrm{End}\, E)$,
h a ~~[illegible]~~ metric on E on D_h the Chern connection of h.
Then $D = D_h + \varphi + \varphi^*$ is flat and h is harmonic on (E, D).]

proof: Not hard (see blackboard).

def A Higgs bundle on X is a holomorphic vector bundle $(E, \bar\partial_E)$ together with a holomorphic one-form $\varphi \in \Omega^{1,0}(\mathrm{End}\, E)$. $(\bar\partial_E \varphi = 0)$

Conclusion Here is where we are at this point:

- Given $\rho: \pi = \pi_1(X) \to G$, we can construct a flat bundle (E, D).
  If we can (??) choose (how many choices?) a harmonic metric on E,
  the we can construct a Higgs bundle out of this. (and h satisfies Hitchin's equations)
- Given a Higgs bundle $(E, \bar\partial_E, \varphi)$, if we can (?) choose (?) a ~~[illegible]~~ metric h that satisfies Hitchin's equations, then we can construct a flat bundle (E, D) (and h is harmonic) and get a representation $\rho: \pi_1(X) \to G$ out of this.

$\boxed{\text{VII Stability}}$ (In the actual talks, I pushed this back to LECTURE 3/4).

$\boxed{\text{VII.1 Representations}}$

Let G be a (reductive?) Lie group, Γ discrete group. (~~Actually $\Gamma = \pi_1(S)$~~)
A representation $\rho: \Gamma \to G$ is called reductive if the Zariski closure of its image is a reductive subgroup of G.

A subgroup $H \subset G$ is called reductive if its Lie algebra $\mathfrak{h}$ is a reductive subalgebra of $\mathfrak{g}$.

$\mathfrak{h} \subset \mathfrak{g}$ is called reductive if the adjoint representation $\mathrm{ad}: \mathfrak{h} \to \mathrm{Aut}(\mathfrak{g})$ is completely reducible.

example : $G = (P)SL_2\mathbb{C}$

ρ not reductive $\Leftrightarrow$ ρ looks like $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ $\Leftrightarrow$ ρ fixes exactly one point in $\mathbb{CP}^1$

Character variety $\quad \Gamma = \pi = \pi_1(S) \qquad G$ algebraic reductive

G acts on $\mathrm{Hom}(\pi, G)$ by conjugation. The quotient is bad

$\to$ one takes the algebraic quotient $\mathrm{Hom}(\pi, G) /\!/ G =: \mathcal{X}(S)$ (GIT quotient)

This is an algebraic variety. irreducible reps form a dense smooth open set.

Prop There is a bijection $\mathcal{X}(S) \longleftrightarrow \mathrm{Hom}^{red}(\pi, G)/G$

where $\mathrm{Hom}^{red}(\pi, G)$ is the subset of reductive representations.

VII. 2 Connections

Let (E, D) be a vectorbundle with a flat connection

def
- (E, D) is called simple if there are no D-invariant subbundles
- (E, D) is called semisimple if it is a direct sum of simple bundles.

VII. 3. Higgs bundles

Degree of a vector bundle

Let E be any complex vector bundle.

Let D be any connection on E (always exists)

def $\deg(E) = \frac{?\,1}{2i\pi} \int_M \mathrm{tr}(F_D)$. $\qquad$ Note : $F_D \in \Omega^2(\mathrm{End}\,E)$

$\mathrm{tr}(F_D) \in \Omega^2(\mathbb{C})$

Lemma This definition does not depend on the choice of D.

proof Let D_1, D_2 connections. $\quad D_2 = D_1 + A \quad A \in \Omega^1(\mathrm{End}\,E)$

$F_{D_2} = F_{D_1} + D_1 A + A \wedge A$ trace-free

$\mathrm{tr}(F_{D_2}) = \mathrm{tr}(F_{D_1}) + \mathrm{tr}(D_1 A)$

$= d(\mathrm{tr}\,A)$.

Note
- If E admits a flat connection, $\deg E = 0$
- In part $\forall \rho \in \mathrm{Hom}(\pi, G)$ $\deg E_\rho = 0$.

Stability of a Higgs bundle

Note: Given a Higgs bundle (E, φ), a Higgs subbundle is a holomorphic φ-invariant subbundle.

def: Let (E, φ) be a degree zero Higgs bundle.

- (E, φ) is called stable if every Higgs subbundle has negative degree.
- (E, φ) is called polystable if it is a direct sum of stable Higgs bundles.

VIII Isomorphisms between moduli spaces (at last)

Here $G = GL(n, \mathbb{C})$ or $SL(n, \mathbb{C})$, say

Rely on two "heavy" theorems:

Theorem (Donaldson, Corlette) Let (E, D) be a flat vector bundle. Then (E, D) admits a harmonic metric $h \iff (E, D)$ is semi-simple.

proof: $\Rightarrow$ not too hard (see e.g. Guichard's Autrans notes)
$\Leftarrow$ hard!

This metric is unique up to gauge equivalence (automorphisms of E, constitute gauge group $\mathcal{G}$)

Theorem (Hitchin-Simpson) Let (E, φ) be a degree zero Higgs bundle. Then (E, φ) admits a metric h satisfying Hitchin's equations $\iff$ (E, φ) is polystable

proof: hard!

This metric is unique up to gauge equivalence.

Let $\mathcal{M}_{Betti}(S, G) = \mathrm{Hom}^{red}(\pi_1(S), G)/_G \simeq \mathcal{X}(S, G)$ "Betti moduli space"

$\mathcal{M}_{dR}(S, G) = \left\{ \begin{array}{l} (E, D) \\ \text{rank } n \text{ flat vector bundle} \\ D \text{ semi-simple} \end{array} \right\} / \mathcal{G}$ "de Rham moduli space"

$\mathcal{M}_{Dol}(X, G) = \left\{ \begin{array}{l} (E, \varphi) \text{ degree zero rank } n \\ \text{polystable Higgs bundle} \end{array} \right\} / \underset{\text{gauge group}}{\mathcal{G}}$ "Dolbeault moduli space" (or Higgs).

It should be clear now that $\boxed{\mathcal{M}_{Betti}(S, G) \simeq \mathcal{M}_{dR}(S, G) \simeq \mathcal{M}_{Dol}(X, G)}$.

This achieves one of the main goals of these talks.

Remarks • Note that the construction of $\mathcal{M}_{Dol}(X, G)$ absolutely requires the choice of a complex structure X on S. The identifications of the other moduli spaces with this one depend on that choice.

• There is a slight subtlety in the definition of $\mathcal{M}_{dR}$: working by components, we ~~fix~~ fix E and let only the flat structure D vary.
Similarly, in $\mathcal{M}_{Dol}$, working by components, we fix E and let only $\bar\partial_E$ (holo str) and φ vary.

• Under these identifications,
ρ irreducible $\Leftrightarrow (E, D)$ simple $\Leftrightarrow (E, \varphi)$ stable
ρ reductive $\Leftrightarrow (E, D)$ semi simple $\Leftrightarrow (E, \varphi)$ polystable.

LECTURE 3/4

In the actual talks, I included most of VII. and VIII here.

IX The Higgs moduli space (very brief overview)

$G = GL(n, \mathbb{C})$ (or $SL(n, \mathbb{C})$) focus $n = \mathrm{rk}\, G$ ($SL(n, \mathbb{C})$ case : $\mathrm{tr}\, \varphi = 0$ where φ is the Higgs field).
X = Riemann surface (closed, genus $g \geq 2$).

As we have defined it, $\mathcal{M}_{Dol}(X, G)$ is just a set. But actually, in a natural way, ...

prop • $\mathcal{M}$ is a complex irreducible quasiprojective algebraic variety of dimension $\dim_{\mathbb{C}} \mathcal{M} = 2n^2(g-1) + 2$ (~~not sure, doesn't seem right~~).
Stable points form a dense smooth open set.

• $\mathcal{M}$ enjoys a hyperkähler structure given by trisymplectic reduction of the natural "hyperkähler" structure on the infinite dimensional vector space $\Omega^{0,1}(M, \mathrm{Ad}\, E) \oplus \Omega^{1,0}(M, \mathrm{Ad}\, E)$, see Hitchin's paper.

• The identification $\mathcal{M}_{Betti} \simeq \mathcal{M}_{Dol}$ is complex symplectic for the appropriate complex structure on $\mathcal{M}_{Dol}$ and Goldman's complex symplectic structure on $\mathcal{M}_{Betti}$,

- but $\mathcal{M}_{\text{Belti}} \simeq \mathcal{M}_{\text{Dol}}$ is not a holomorphic ~~~~ identification for the standard complex structure on $\mathcal{M}_{\text{Dol}}$ (which comes from the complex structure X on S, whereas the complex structure on $\mathcal{M}_{\text{Belti}}$ comes from the complex structure on G).

- $\mathcal{M}$ comes equipped with a $U(1)$ (or $\mathbb{C}^*$) action : $\lambda.(E,\varphi) = (E, \lambda\varphi)$

What does this action mean in terms of geometric structures ?

- $\mathcal{M}$ comes equipped with a Hitchin function $f : \mathcal{M} \longrightarrow \mathbb{R}$, $(E,\varphi) \longmapsto$ energy of harmonic metric

f is a moment map for the $\mathbb{C}^*$ action.

$\boxed{\text{X Reality}}$ [motivation : cf lower]

Quick overview of the notion of complexification :

- Of a vector space $V \rightsquigarrow V^{\mathbb{C}} = V \otimes \mathbb{C} = V \oplus iV$ comes with a conjugation $\tau \in \mathrm{End}_{\mathbb{R}}($
  + complexification of linear maps.
- Of a Lie algebra : same.
- Of a Lie group. Note : $\underbrace{U(n)^{\mathbb{C}} = GL(n,\mathbb{R})^{\mathbb{C}}}_{\text{real forms}} = GL(n,\mathbb{C})$
- Of a vector bundle, of tensors...

Note: Complexifying a complex object (e.g vector space) does not leave it unchanged !

prop : let $E \to M$ real vector bundle, $E^{\mathbb{C}} \to M$ complexification.

- h Hermitian metric comes from Riemannian metric on $E \iff \overline{h(s_1,s_2)} = h(\bar{s}_1, \bar{s}_2)$
- D connection on $E^{\mathbb{C}}$ comes from real connection on $E \iff \forall s \ \overline{Ds} = D\bar{s}$.

"h is compatible"

- motivation :

So far we have looked at representations in $G = GL(n,\mathbb{C})$ (or $SL(n,\mathbb{C})$) say, and G - Higgs bundles, G flat bundles and so forth.

What if we want to consider real representations, say $G = SL(n,\mathbb{R})$?

Two (equivalent) strategies are possible :

- Simply observe that $\mathcal{X}(S, SL_n\mathbb{R}) \subset \mathcal{X}(S, SL_n\mathbb{C})$ and try to identify Higgs bundles which correspond to real representations (not necessarily easy)
- Define a moduli space of real Higgs bundles]

prop Let (E, D) be a real flat semi-simple vector bundle, $(E^{\mathbb{C}}, D^{\mathbb{C}})$ complexifica
Then the harmonic metric on $E^{\mathbb{C}}, D^{\mathbb{C}}$ is compatible.

Let $E^{\mathbb{C}}$ be the complexification of a real vector bundle, h ~~harmonic~~ compatible metric.
Then $Q(s_1, s_2) := h(s_1, \bar{s}_2)$ is a nondegenerate complex bilinear form on $E^{\mathbb{C}}$.

Conversely, let F be a complex vector bundle, h Hermitian metric, Q nondegenerate complex bilinear form on F. Then $\exists!$ "conjugation" $\tau : F \to F$ such that $Q(s_1, s_2) = h(s_1, \tau s_2)$. τ identifies $F \simeq E^{\mathbb{C}}$ (where E = fixed points of τ) and h is compatible (qu: Is it clear that $\tau^2 = id$?)

def Let $G = GL(n, \mathbb{R})$, $G^{\mathbb{C}} = GL(n, \mathbb{C})$.
A G-Higgs bundle is (F, φ, Q) where:
- (F, φ) is a $(G^{\mathbb{C}})$ Higgs bundle
- Q is a holomorphic nondegenerate complex bilinear form on F
- φ is Q-symmetric.

prop Let $\mathcal{M}_{Dol}^{\mathbb{R}}(X, G_{\mathbb{R}})$ be the moduli space of degree zero polystable $G_{\mathbb{R}}$-Higgs bundle on X.
Riemann surface

Then $\boxed{\mathcal{M}_{Betti}(S, G_{\mathbb{R}}) \approx \mathcal{M}_{dR}(S, G_{\mathbb{R}}) \approx \mathcal{M}_{Dol}^{\mathbb{R}}(X, G_{\mathbb{R}})}$
(and these identifications respect the inclusions induced by $G_{\mathbb{R}} \subset G_{\mathbb{C}}$)

XI Hitchin fibration

Let (E, φ) Higgs bundle on Riemann surface X.
$K = (T^{1,0})^* X$ canonical bundle on X (holomorphic line bundle)

φ is a holo. $(1,0)$ form with values in End $E \Leftrightarrow \varphi$ holo. bundle map $E \to E \otimes K$

One can construct
$$\varphi^{\otimes k} : E^{\otimes k} \longrightarrow E^{\otimes k} \otimes K^{\otimes k} \quad (K^{\otimes k} = K^k)$$
$$\Lambda^k \varphi : \Lambda^k E \longrightarrow \Lambda^k E \otimes K^k$$

i.e. $\Lambda^k \varphi \in H^0(\mathrm{End}(\Lambda^k E) \otimes K^k)$
holo. section

and $\mathrm{tr}(\Lambda^k \varphi) \in H^0(K^k)$.

Moreover, this is invariant under the action of the gauge group, so one can define the Hitchin fibration

$$p : \underbrace{\mathcal{M}^{s}_{Dol}}_{\text{stable part}}(X, \underset{\substack{GL(n,\mathbb{R})}}{G}) \longrightarrow T := \bigoplus_{k=1}^{n} H^0(X, K^k).$$

Dimension of T: $\dim_{\mathbb{C}} H^0(X, K) = g$
$\dim_{\mathbb{C}} H^k(X, K^2) = (2k-1)(g-1) \quad (k>0)$

$$\to \dim_{\mathbb{R}} T = \dim_{\mathbb{R}} \mathcal{M}^{s}_{Dol}(X, G) = \tfrac{1}{2} \dim_{\mathbb{R}} \mathcal{M}^{s}_{Dol}(X, GL_n \mathbb{C}) = n^2(g-1)+$$

The Hitchin section $\exists\, s : T \longrightarrow \mathcal{M}^{\mathbb{R}\, s}_{Dol}(X, GL(n,\mathbb{R}))$ such that s is injective and s is a section to p (up to an automorphism of T).
s is explicit: cf Hitchin's paper or Guichard's Autrans notes.

Thm: $s(T) \subset \mathcal{M}^{\mathbb{R}\, s}_{Dol}(X, GL(n,\mathbb{R}))$ is a connected component (in fact, the Hitchin component, cf later).

Proof: $s(T)$ is open by injectivity + dimensions (invariance of domain)
$s(T)$ is closed since it is a section to p
($s(0)$ is Fuchsian $\to$ Hitchin component).

LECTURE 4/4

$\boxed{\text{XII The Hitchin component}}$

The $PSL(2,\mathbb{R})$ character variety (see also next section)

Every representation $\rho : \pi_1(S) \longrightarrow G$ Lie group has a characteristic class $c(\rho) = c(E_\rho) \in \pi_1(G$
Here $G = PSL(2,\mathbb{R})$ $\pi_1(G) = \mathbb{Z}$ $e(\rho) = c(\rho) \in \mathbb{Z}$ is the Euler class of ρ.

Thm (Goldman) The components of $\mathcal{X}(S, PSL_2\mathbb{R})$ are classified by the Euler class of representations which takes all values in $[\![-(g-1), \dots, g-1]\!]$ (g = genus of S)

Thm (Goldman) continued: Maximal representations in the sense that $|e(\rho)| = g-1$ are the discrete and faithful representations, they are the holonomy of hyperbolic structures on S.

let $\mathcal{F}(S) := \{[\rho] \in \mathcal{X}(S),\ e(\rho) = g-1\}$ Fuchsian or Fricke space of S:

$\mathcal{F}(S) \simeq \left\{ \begin{smallmatrix} \text{complete} \\ \text{hyperbolic} \\ \text{structures on } S \end{smallmatrix} \right\} / \text{isotopy}$ (deformation space of hyperbolic structures on S)

The Teichmüller space of S is the deformation space of complex (or conformal) structures on S:

$\mathcal{T}(S) = \left\{ \begin{smallmatrix} \text{complex structures} \\ \text{on } S \end{smallmatrix} \right\} / \text{isotopy}$

There is a map $\left\{ \begin{smallmatrix} \text{hyperbolic structures} \\ \text{on } S \end{smallmatrix} \right\} \longrightarrow \left\{ \begin{smallmatrix} \text{conformal structures} \\ \text{on } S \end{smallmatrix} \right\}$: just take the conformal class of a complete hyperbolic metric.

By the uniformization theorem (nontrivial), this map induces an isomorphism $\mathcal{F}(S) \xrightarrow{\sim} \mathcal{T}(S)$.

<u>The principal $\mathfrak{sl}_2(\mathbb{R})$</u>

let $\mathfrak{g}$ be a real split semisimple Lie algebra, $\mathfrak{g}$ contains a "principal $\mathfrak{sl}_2\mathbb{R}$".

Let G be a real split semi-simple Lie group, there is a unique (?) irreducible representation $i: PSL_2(\mathbb{R}) \longrightarrow G$.

If $G = PSL_n(\mathbb{R})$, i is given by the action of $PSL_2(\mathbb{R})$ on $S^{n-1}\mathbb{R}^2$. (symmetric power)

(Let $V = \left\{ \begin{smallmatrix} P \text{ homogeneous polynomial} \\ \text{of degree } n-1 \text{ in } X, Y \end{smallmatrix} \right\}$ $PSL_2(\mathbb{R}) \circlearrowright V$ by "$P \mapsto P(aX+bY, cX+dY)$)

<u>Fuchsian representations, Hitchin component</u>

A representation $\rho: \pi_1(S) \longrightarrow G$ is called <u>Fuchsian</u> if it factors $\rho: \pi_1(S) \xrightarrow{\rho_0} PSL_2(\mathbb{R}) \xrightarrow{i} G$ with ρ_0 Fuchsian in the classical sense.

The Hitchin component is the connected component of $\mathcal{X}(S, G)$ which contains the classes of Fuchsian representations.

<u>Thm (Hitchin)</u> The Hitchin component is topologically a cell of dimension $2n^2(g-1)+2$ where $n = \mathrm{rk}\, G$ $\quad g = $ genus of S.

proof: sketched in previous section. (This was one of the main goals of these talks).

The Hitchin component is our ~~th~~ higher Teichmüller space. Corresponding representations are called Hitchin representations. Do they have nice properties, similar to Fuchsian representations?

<u>Thm (Hitchin)</u> If $n > 2$, the number of connected components of $\mathcal{X}(S, PSL_n(\mathbb{R}))$ is 3 if n is odd, 6 if n is even.

- Let us mention the following theorems:

• Introducing the notion of Anosov representations $\rho: \pi_1(S) \to PSL(n,\mathbb{R})$, Labourie proved:

Thm: Hitchin representations are discrete and faithful (and reductive).

He also proved: Thm A Hitchin rep. is diagonalizable with real distinct eigenvalues.

• This characterization is similar to the characterization of maximal representations:

Thm: let $\rho: \pi_1(S) \to PSL(n,\mathbb{R})$. Then ρ is Hitchin if and only if there exists a ρ-equivariant convex curve $\xi: \partial\pi \to \mathbb{R}P^{n-1}$.
(Labourie-Guichard)

In part. (?), Hitchin representations are Anosov.

• Thm (Goldman-Choi) Hitchin representations $\rho: \pi_1(S) \to PSL_3(\mathbb{R})$ are the holonomy representations of real convex projective structures on S.

XIII Example: SL(2)

Let $G = \mathbb{P}SL(2,\mathbb{C})$. $\mathcal{M}_{Dol}(X, G)$ moduli space of Higgs bundles.
(X: Riemann surface)

Can we identify Higgs bundles associated to real representations?

The real forms of G are $SU(2)$ and $SL(2,\mathbb{R})$.

On the level of Lie algebras, the real form $su(2)$ is associated to the conjugation $\tau: A \mapsto -A^*$
• the real form $sl(2,\mathbb{R})$ is associated to the conjugation $\sigma: A \mapsto A^*$.

τ and σ are inner equivalent: $\sigma = J\tau J^{-1}$ where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in sl(2,\mathbb{R})$
so they induce the same antiholomorphic involution i on $\mathcal{X}(S,G)$.

prop Under the identification $\mathcal{X}(S,G) \approx \mathcal{M}_{Dol}$, i corresponds to the involution
$j: \mathcal{M}_{Dol} \to \mathcal{M}_{Dol}$, $(E,\varphi) \mapsto (E,-\varphi)$.

(proof: $D = \tau(\bar\partial_E) + \bar\partial_E + \varphi - \tau(\varphi) \rightsquigarrow \tau(D) = \bar\partial_E + \tau(\bar\partial_E) + \tau(\varphi) - \varphi$.)

prop The fixed points of j are of two types:

(1) $(E,\varphi) \in \mathcal{M}_{Dol}$, $\varphi = 0$. These correspond to $SU(2)$ representations
$\rightsquigarrow$ already known by a theorem of Narasimhan-Seshadri.

(2) $(E,\varphi) \in \mathcal{M}_{Dol}$, $\exists L$ holomorphic line bundle such that $E = L \oplus L^{-1}$
and $\varphi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}$ (Note: $\beta \in H^0(X, L^2 K)$ $\gamma \in H^0(X, L^{-2}K)$).

In this case $(E,\varphi) \approx (E,-\varphi)$ under the gauge transformation $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$.

Prop (2) corresponds to $\mathfrak{sl}(2,\mathbb{R})$ representations.

proof: let h be the solution to Hitchin's equations. Then h must preserve $E = L \oplus L^{-1}$
$\rightsquigarrow$ $h = h_L \oplus h_{L^*}$ where h_L is a $U(1)$ connection on L.
$(L^* \approx L^{-1})$

proof 1: $L \oplus L^{-1}$ $U(1)$ bundle $\Leftrightarrow$ $L \oplus L^{-1} = L_{\mathbb{R}} \otimes \mathbb{C}$ where $L_{\mathbb{R}}$ $SO(2)$ bundle.

proof 2: In local chart U_α $D_{\bar{\partial}_E, h} = d + A_\alpha$ $A_\alpha = \begin{pmatrix} A_{\alpha,L} & 0 \\ 0 & A_{\alpha,L^*} \end{pmatrix}$

with $A_{\alpha,L} = i\,a_\alpha \in \Omega^1(U_\alpha,\mathbb{R})$ $A_{\alpha,L}^* = -i a_\alpha$.

and $\varphi + \varphi^* = \begin{pmatrix} 0 & \varphi_\alpha \\ \overline{\varphi_\alpha} & 0 \end{pmatrix}$.

So $D = d + \begin{pmatrix} i a_\alpha & \varphi_\alpha \\ \overline{\varphi_\alpha} & -i a_\alpha \end{pmatrix}$; this matrix lies in the copy of $\mathfrak{sl}(2,\mathbb{R}) \subset \mathfrak{sl}(2,\mathbb{C})$ obtained by conjugation by $\begin{pmatrix} 1 & i \\ i & -i \end{pmatrix}$.

Note: $\gamma \in H^0(L^{-2}K)$ $\gamma \neq 0$ (stability) $\rightarrow$ $\deg(L^{-2}K) \geq 0$ (admits nonzero holo section)
$\rightarrow$ $\deg L \leq g-1$ Milnor-Wood inequality.

maximal reps: $\deg L = g-1$ $\Rightarrow$ $L^2 = K$.
In this case, γ is a constant $\rightarrow$ rescale $\gamma = 1$.
the Higgs field is $\varphi = \begin{pmatrix} 0 & \beta \\ 1 & 0 \end{pmatrix}$ with $\beta \in H^0(K^2)$: β holomorphic quadratic differential.

This gives a parametrization of maximal reps (soon seen to be Fuchsian space) by $H^0(K^2)$.

For $\beta = 0$, the Hitchin equations become $F_h = -2\,\mathrm{dvol}$
So the solution of Hitchin's equations is a constant curvature metric, so it comes from a Fuchsian representation.

Note: In particular, we get back the uniformization theorem in this extremely simple case. This shows the deepness of Hitchin (-Simpson)'s theorem.