Minimal surfaces and quasi-Fuchsian structures

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Abstract

These are the notes written after my talk in the workshop *Higgs bundles and harmonic maps* that was held in Asheville, NC in Januray 2015, organized by Brian Collier, Qiongling Li and Andy Sanders and supported by the NSF GEAR Network.

We review aspects of the theory of minimal surfaces in hyperbolic 3-manifolds and their importance in the study of representations of surface groups into $PSL_2(\mathbb{C})$ and related deformation spaces, such as the deformation space of quasi-Fuchsian structures $\mathcal{QF}(S)$, Taubes' moduli space of minimal hyperbolic germs $\mathcal{H}$ and the moduli space of Higgs bundles $\mathcal{M}$.

Contents

Introduction

In preparation.

1 Minimal surfaces

In this section we review some basics of the theory of minimal surfaces. We start from the general setting of harmonic maps between Riemannian manifolds and gradually specialize to minimal surfaces in hyperbolic 3-manifolds, highlighting the specific features that appear in the process. References for this section include [12] and [8].

Prepare the way for later.

1.1 Harmonic and minimal maps between Riemannian manifolds

Consider smooth Riemannian manifolds (M, g) and (N, h) and a smooth map $f : M \to N$.

Vector-valued second fundamental form and tension field

The derivative df can be described as a section of the bundle $E := T^*M \otimes f^*TN \to M$. This bundle is equipped with a connection ∇ induced from the Levi-Civita connections ∇^g and ∇^h of g and h. Hence one can take the covariant derivative $\nabla(df)$, it is a section of $T^*M \otimes T^*M \otimes f^*TN$. This tensor turns out to be symmetric in the first two factors (see below).

Definition 1.1. The *(vector-valued) second fundamental form* of f is $\underline{\mathrm{II}}(f) = \nabla(df)$[1], seen as a symmetric covariant 2-tensor on M with values in f^*TN.

By definition of ∇ (enforcing the Leibniz rule), $\underline{\mathrm{II}}(f)$ is given by

$$\underline{\mathrm{II}}(f)(X, Y) = (f^*\nabla^h)_X(df(Y)) - df(\nabla^g_X Y) \tag{1.1}$$

for vector fields X, Y on M. Abusing notations, we can write $\underline{\mathrm{II}}(f) = \nabla^h \circ df - df \circ \nabla^g$. It appears now that the symmetry of $\underline{\mathrm{II}}_f$ follows from the fact that ∇^h and ∇^g are torsion-free. When f is an immersion so that M is thought of as an immersed submanifold of N (equipped with a different metric), we let ourselves maltreat notations further and write

$$\underline{\mathrm{II}}(f) = \nabla^h{}_{|M} - \nabla^g\,. \tag{1.2}$$

Here is an example of a first use we can make of this definition: recall that f is called a *totally geodesic map* is it sends geodesics of (M, g) into geodesics of (N, h). Then

[1] We use the underline notation $\underline{\mathrm{II}}$ for the vector-valued second fundamental form in order to distinguish it from the real-valued second fundamental form II introduced in definition 1.8.

Proposition 1.2. *$f : M \to N$ is a totally geodesic map if and only if $\underline{\mathrm{II}}(f)$ vanishes identically.*

Next we define the tension field[2] of f:

Definition 1.3. The *tension field* $\tau(f)$ is the section of f^*TN obtained by contracting $\underline{\mathrm{II}}(f)$ using the metric g ("taking the trace"): $\tau(f) = \mathrm{tr}_g\, \underline{\mathrm{II}}(f)$.

Harmonic maps

We can now define harmonicity of maps between Riemannian manifolds:

Definition 1.4. $f : M \to N$ is called *harmonic* if it has everywhere vanishing tension field.

Remark 1.5. Here are a couple useful remarks about harmonic maps between Riemannian manifolds:

- The tension field $\tau(f)$ can be interpreted as the gradient of the energy functional $\mathcal{E}(f) = \frac{1}{2}\int_M \|df\|^2_{g,h}\, dvol_g$. Accordingly, a harmonic map is classically defined as a critical point of the energy funtional $\mathcal{E}$.
- One can check that f is a harmonic map if and only if df is a harmonic one-form with values in f^*TN in the sense of Hodge theory.
- It can be showed that f is harmonic if and only if it preserves centers of mass infinitesimally, in an appropriate sense.

Minimal immersions

Finally we define minimal maps between Riemannian manifolds:

Definition 1.6. $f : M \to N$ is called *minimal* if it is an isometric harmonic map.

Recall that an isometric map is necessarily an immersion. Naturally, if M is not equipped with a Riemannian metric, an immersion $f : M \to (N, h)$ is called minimal if it is a harmonic map from (M, g) to (N, h) where $g = f^*h$ is the pull-back metric on M.

Let us make a couple of remarks before specializing to minimal hypersurfaces:

Remark 1.7.

- When $f : M \to N$ is an isometric map, the energy element $\|df\|^2_{g,h}\, dvol_g$ (see Remark 1.5) is just the area density of the immersion $M \to N$. Accordingly, a minimal map is classically defined as a critical point of the area functional.
- The study of minimal surfaces goes back to Euler and Lagrange who gave their names to the Euler-Lagrange equation (which here is just $\mathrm{II}(f) = 0$). The typical physical model of a minimal surface in Euclidean space is obtained by dipping a wire frame into a soap solution. Showing the existence of a minimal surface with given boundary is called *Plateau's problem*; mathematician Jesse Douglas was awarded the Fields medal for solving it in 1936.

[2]The tension field could also be called the *vector-valued mean curvature*.

1.2 Minimal hypersurfaces

When $f : (M, g) \longrightarrow (N, h)$ is isometric, one can check that $\underline{\mathrm{I\!I}}(f)$ is always orthogonal to $f_*(TM)$. An important particular case is when f is a two-sided hypersurface immersion; in this situation $\underline{\mathrm{I\!I}}(f)$ is completely determined by the real-valued second fundamental form:

Definition 1.8. Let $f : M \longrightarrow N$ be an isometric hypersurface immersion. Assume that it is two-sided so that one can fix a choice[3] of a unit normal vector field n to M. The *real-valued second fundamental form*[4] of f is the quadratic form $\mathrm{I\!I}(f)$ on M such that $\underline{\mathrm{I\!I}}(f) = \mathrm{I\!I}(f)\, n$.

It is clear that $\mathrm{I\!I}(f)$ is given by $\mathrm{I\!I}(f)(X, Y) = h(\underline{\mathrm{I\!I}}(f)(X, Y), n)$ for vector fields X, Y on M. Looking back at (1.2), that is $\mathrm{I\!I}(f)(X, Y) = h(\nabla^h_X Y, n)$. Since ∇^h is the Riemannian connection of h, this can also be written:

Proposition 1.9. *The real-valued second fundamental form of f is given by*

$$\mathrm{I\!I}(f)(X, Y) = h(\nabla^h_X Y, n) = -h(\nabla^h_X n, Y) \tag{1.3}$$

for any two vector fields X, Y on M.

Definition 1.10. Let $f : M \longrightarrow N$ be a two-sided hypersurface isometric immersion. We define the following classical *extrinsic invariants*:

- The *first fundamental form* $\mathrm{I}(f)$ is just the metric on M: $\mathrm{I}(f) = g = h_{|M}$.
- The *shape operator* $B(f)$ is the g-self-adjoint[5] endomorphism of TM associated to $\mathrm{I\!I}(f)$.
- The *mean curvature*[6] $H(f)$ is function on M defined by $H(f) = \mathrm{tr}(B(f)) = \mathrm{tr}_g(\mathrm{I\!I}(f))$.
- The *principal curvatures* $\lambda_k(f)$ are the eigenvalues of the shape operator[7], they are functions on M.

By definition of the mean curvature, the tension field of such a map is given by $\tau(f) = H(f)\, n$, it follows that

Proposition 1.11. *Let $f : M \longrightarrow N$ be a two-sided hypersurface isometric immersion. Then f is minimal if and only if it has everywhere vanishing mean curvature.*

1.3 Minimal surfaces

A straightforward observation is that when $\dim M = 2$, the energy element $\|df\|^2_{g,h}\, dvol_g$ (see Remark 1.5) is invariant under a conformal change of the metric on M ($g \longrightarrow e^{2u} g$). It follows that

[3]Note that if both M and N are orientable then f is necessarily two-sided, and a choice of a unit normal vector field is determined by a choice of orientation of both M and N.

[4]The opposite sign convention for $\mathrm{I\!I}(f)$ is sometimes preferred.

[5]This means that $B(f)$ is characterized by $g(B(f)(X), Y) = \mathrm{I\!I}(f)(X, Y)$. It follows from (1.3) that $B(f)$ is given by $B(f)(X) = -\nabla^h_X n$. For this reason the opposite sign convention for $B(f)$ is often preferred.

[6]The convention $H(f) = \pm\frac{\mathrm{tr}(B(f))}{\dim M}$ is classically preferred.

[7] Recall that by the spectral theorem, the quadratic form $\mathrm{I\!I}(f)$ can be diagonalized in a g-orthonormal basis, in this sense the principal curvatures can also be defined as the eigenvalues of $\mathrm{I\!I}(f)$.

Proposition 1.12. *If* $\dim M = 2$*, the harmonicity of a map* $f : (M, g) \to (N, h)$ *only depends on the conformal class of* g*.*

Therefore if $M = S$ is a surface equipped with a conformal structure, then it makes sense to talk about harmonic maps from S to a Riemannian manifold (N, h). Recall that provided S comes with an orientation, a conformal structure on S is equivalent to a complex structure on S compatible with the orientation, in other words a Riemann surface structure. In these notes we shall often denote by X a surface S equipped with a Riemann surface structure. It is also clear that

Proposition 1.13. *Let* X *be a Riemann surface and* (N, h) *a Riemannian manifold. A minimal map* $f : X \to N$ *is a conformal harmonic map from* X *to* N*.*

Hopf differential

Definition 1.14. Let $f : X \to N$ where X is a Riemann surface and (N, h) a Riemannian manifold. The *Hopf differential* of f is the $(2, 0)$-part of the pull-back of the metric h on X: $\mathrm{Hopf}(f) = (f^*h)^{(2,0)}$.

Thus $\mathrm{Hopf}(f)$ is a complex quadratic differential on X, but it is not necessarily holomorphic. The following proposition is straightforward to prove and will be used later:

Proposition 1.15. *Let* $f : X \to N$ *where* X *is a Riemann surface and* (N, h) *a Riemannian manifold.*
(i) If f *is harmonic then* $\mathrm{Hopf}(f)$ *is holomorphic.*
(ii) f *is conformal if and only if* $\mathrm{Hopf}(f) = 0$.

1.4 Minimal surfaces in hyperbolic 3-manifolds

Now $(M, g) = (S, g)$ is a surface with a Riemannian metric and $(N, h) = \mathbb{H}^3$ is hyperbolic 3-space, or more generally (N, h) is a hyperbolic 3-manifold (*i.e.* $\dim N = 3$ and h has constant sectional curvature -1).

Gauss-Codazzi equations

The classical Gauss-Codazzi equations for immersed submanifolds are expressed in this setting as follows:

Proposition 1.16. *Let* $f : S \to N$ *be a two-sided immersion. Then* $\mathrm{I} = g$ *and* $\mathrm{II} = \mathrm{II}(f)$ *satisfy the Gauss-Codazzi equations:*

$$\begin{cases} K_g = -1 + \det{}_g \mathrm{II} & \textit{(Gauss equation)} \quad (1.4) \\ d_{\nabla^g} \mathrm{II} = 0 & \textit{(Codazzi equation)} \quad (1.5) \end{cases}$$

where
- K_g *is the curvature of* g*,*

- $\det_g \mathbb{I} = \lambda_1\lambda_2$ *where* λ_k *are the principal curvatures*[8],
- $d_{\nabla^g} : \Omega^1(T^*S) \longrightarrow \Omega^2(T^*S)$ *is the extension of the exterior derivative to differential forms with values in* T^*S *(using the Levi-Civita connection* ∇^g*).*

Conversely, the "fundamental theorem of surface theory"[9] states that the choice of any pair $(g, \mathbb{I})$ on S satisfying the Gauss-Codazzi equations uniquely determines an immersion of S into a possibly incomplete hyperbolic 3-manifold N. The unicity of N is up to isometry and up to restricting to a neighborhood of the immersed surface, in other words it is the *germ* of N that is unique. Under this clarification:

Theorem 1.17.a. *Let S be a smooth orientable surface. Let g be a Riemannian metric on S and $\mathbb{I}$ a quadratic form such that the Gauss-Codazzi equations* (1.4) *and* (1.5) *are satisfied. Then there exists a unique immersion f of S into the germ of a* 3*-dimensional hyperbolic thickening N such that* $\mathrm{I}(f) = g$ *and* $\mathbb{I}(f) = \mathbb{I}$.

Note that when N is complete, then $N = \mathbb{H}^3/\Gamma$ where Γ is a Kleinian group (see section 2.1) and lifting to universal covers produces an equivariant immersion $\tilde{f} : \tilde{S} \longrightarrow \mathbb{H}^3$. But even when N is incomplete, its hyperbolic structure defines a *developing map* $\mathrm{dev}_N : \tilde{N} \longrightarrow \mathbb{H}^3$ and a *holonomy representation* $\rho_N : \pi_1 N \longrightarrow \mathbb{H}^3$ such that dev_N is ρ_N-equivariant. The pair (dev_N, ρ_N) is unique up to the action of $\mathrm{PSL}_2(\mathbb{C})$ by post-composition on dev_N and conjugation on ρ_N[10]. We thus get an immersion $\mathrm{dev}_N \circ \tilde{f} : \tilde{S} \longrightarrow \mathbb{H}^3$ that is ρ-equivariant for $\rho = \rho_N \circ f_* : \pi_1 S \longrightarrow \mathrm{PSL}_2(\mathbb{C})$. It is easy to check that given a ρ-equivariant immersion $\tilde{S} \longrightarrow \mathbb{H}^3$, there is no other ρ satisfying equivariance. Putting all this together, let us give a second version of the previous theorem:

Theorem 1.17.b. *Let S be a smooth orientable surface. Let g be a Riemannian metric on S and $\mathbb{I}$ a quadratic form such that the Gauss-Codazzi equations* (1.4) *and* (1.5) *are satisfied. Then there exists a ρ-equivariant immersion $f : \tilde{S} \longrightarrow \mathbb{H}^3$ such that* $\mathrm{I}(f)$ *and* $\mathbb{I}(f)$ *are the lifts of* I *and* $\mathbb{I}$ *to $\tilde{S}$. The pair (f, ρ) is unique up to the action* $\mathrm{PSL}_2(\mathbb{C})$ *by post-composition on f and conjugation on ρ.*

Note that when $f : S \longrightarrow N$ is a minimal immersion *i.e.* $H = \mathrm{tr}_g(\mathbb{I}) = 0$, then the principal curvatures satisfy $\lambda_1 + \lambda_2 = 0$. in particular $\det_g \mathbb{I} = -\lambda_1^2 = -\frac{1}{2}(\lambda_1^2 + \lambda_2^2) = -\frac{1}{2}\| \mathbb{I} \|_g^2$. Hence we define the *Gauss-Codazzi equations for a minimal surface*:

$$\begin{cases} K_g = -1 - \dfrac{1}{2}\| \mathbb{I} \|_g^2 & (1.6) \\ d_{\nabla^g} \mathbb{I} = 0 & (1.7) \\ \mathrm{tr}_g(\mathbb{I}) = 0 & (1.8) \end{cases}$$

and, following Taubes [47], we define a minimal hyperbolic germ:

Definition 1.18. Let S be a smooth orientable surface. A *minimal hyperbolic germ* on S is a couple $(g, \mathbb{I})$ where g is a Riemannian metric on S and $\mathbb{I}$ a quadratic form such that the Gauss-Codazzi equations for a minimal surface are satisfied.

[8] $\det_g \mathbb{I}$ is called the *extrinsic curvature*, as opposed to the "intrinsic curvature" K_g.

[9] This theorem is classically stated for immersed surfaces in Euclidean 3-space and also holds for immersed submanifolds in a Euclidean space of higher dimension, see [24].

[10] One may refer to Thurston's book [49, pages 139-141] for developing maps and holonomy of geometric structures.

We get a “fundamental theorem of surface theory” for minimal surfaces:

Theorem 1.19.a. *Let S be a smooth orientable surface and let $(g, \mathrm{I\!I})$ be a minimal hyperbolic germ on S. Then there exists a unique minimal immersion f of S into the germ of a hyperbolic 3-manifold N such that $\mathrm{I}(f) = g$ and $\mathrm{I\!I}(f) = \mathrm{I\!I}$.*

Let us also give the second version:

Theorem 1.19.b. *Let S be a smooth orientable surface and let $(g, \mathrm{I\!I})$ be a minimal hyperbolic germ on S. Then there exists a ρ-equivariant minimal immersion $f : \tilde{S} \to \mathbb{H}^3$ such that $\mathrm{I}(f)$ and $\mathrm{I\!I}(f)$ are the lifts of I and $\mathrm{I\!I}$ to $\tilde{S}$. The pair (f, ρ) is unique up to the action $\mathrm{PSL}_2(\mathbb{C})$ by post-composition on f and conjugation on ρ.*

The following key observation highlighted by Donaldson [10] and Taubes [47] goes back to Hopf [21]:

Proposition 1.20. *Let S be a smooth oriented surface. Let g be a Riemannian metric and $\mathrm{I\!I}$ a real quadratic form on S. Denote by X the Riemann surface structure on S given by the conformal class of g.*

(i) $\mathrm{I\!I}$ is the real part of a complex quadratic differential α on X if and only if $\mathrm{tr}_g(\mathrm{I\!I}) = 0$[11].

(ii) If (i) holds, then α is a holomorphic quadratic differential on X and only if the Codazzi equation $d_{\nabla^g} \mathrm{I\!I} = 0$ (1.5) holds.

Let us point out that in spite of appearances, this observation is not directly related to proposition 1.15.

2 Quasi-Fuchsian and almost-Fuchsian structures

2.1 Quasi-Fuchsian structures

Let us briefly review quasi-Fuchsian structures. We warn that there are subtleties about quasi-Fuchsian structures that are overlooked in these notes, especially the relation between the quasiconformal theory of surfaces and 3-dimensional hyperbolic geometry[12].

Kleinian groups and complete hyperbolic 3-manifolds

Recall that the Lie group $\mathrm{PSL}_2(\mathbb{C})$ is both the group of automorphisms of the complex projective line $\mathbb{C}\mathbf{P}^1$ (acting by projective linear transformations, a.k.a. Möbius transformations of the Riemann sphere) and the group of orientation-preserving isometries of hyperbolic 3-space $\mathbb{H}^3$. In fact, $\mathbb{C}\mathbf{P}^1$ is the natural “boundary at infinity” of $\mathbb{H}^3$, and the action of $\mathrm{PSL}_2(\mathbb{C})$ on $\mathbb{H}^3$ is the unique isometric action extending the action of $\mathrm{PSL}_2(\mathbb{C})$ on $\mathbb{C}\mathbf{P}^1 = \partial_\infty \mathbb{H}^3$.

[11] A such α is unique, in fact α is necessarily twice the $(2, 0)$-part of $\mathrm{I\!I}$.

[12] A thorough reference for the relation between Kleinian groups and hyperbolic manifolds is [25].

A *Kleinian group* is a discrete torsion-free subgroup of $\mathrm{PSL}_2(\mathbb{C})$. Any Kleinian group Γ automatically acts freely and properly on hyperbolic 3-space $\mathbb{H}^3$, so that the quotient $M := \mathbb{H}^3/\Gamma$ is a complete hyperbolic 3-manifold. Conversely, any complete hyperbolic 3-manifold M can be obtained as $M = \mathbb{H}^3/\Gamma$ where Γ is a Kleinian group, that is the image of the *holonomy* of the hyperbolic structure $\rho : \pi_1 M \to \mathrm{Isom}^+(\mathbb{H}^3)$. However Γ does not act freely and properly on the whole complex projective line $\mathbb{C}\mathbf{P}^1$; the largest open set $\Omega(\Gamma) \subset \mathbb{C}\mathbf{P}^1$ having this property is called the *domain of discontinuity* of Γ and its complement $\Lambda(\Gamma) := \mathbb{C}\mathbf{P}^1 \setminus \Omega(\Gamma)$ is called the *limit set*[13] of Γ. The possibly disconnected surface $\Omega(\Gamma)/\Gamma$ comes equipped with a Riemann surface structure[14] and is the natural "conformal boundary at infinity" $\partial_\infty M$ of the hyperbolic 3-manifold $M = \mathbb{H}^3/\Gamma$. The hyperbolic structure of M is called *convex cocompact* if $\partial_\infty M$ compactifies M (when that happens $M \cup \partial_\infty M$ is topologically the end compactification of M). Equivalently (and more classically), convex cocompactness can be defined as the property that the convex core of M is a compact deformation retract of M. The *convex core* of M is $C(\Lambda)/\Gamma \subset M$, where $C(\Lambda)$ is the convex hull in $\mathbb{H}^3$ of the limit set Λ.

Quasi-Fuchsian groups and quasi-Fuchsian 3-manifolds

We may now define quasi-Fuchsian groups and quasi-Fuchsian 3-manifolds:

Definition 2.1. A Kleinian group Γ is called a *quasi-Fuchsian group* if its limit set Λ is a Jordan curve[15] and if Γ preserves each component of its domain of discontinuity $\Omega = \mathbb{C}\mathbf{P}^1 \setminus \Lambda$.

Note that by Jordan's theorem Ω actually consists of two connected components Ω^+ and Ω^- that are topological disks.

In these notes we will restrict the definition of quasi-Fuchsian groups from now on to only consider groups that are isomorphic to closed surface groups.

Definition 2.2. A *quasi-Fuchsian* 3*-manifold* is a complete convex cocompact hyperbolic 3-manifold that is smoothly diffeomorphic to a product $S \times \mathbb{R}$ where S is a closed connected surface.

The two notions coincide under the correspondence between Kleinian groups and complete 3-dimensional hyperbolic structures discussed above:

Proposition 2.3. *A complete hyperbolic* 3*-manifold* $M = \mathbb{H}^3/\Gamma$ *is quasi-Fuchsian if and only if the Kleinian group* Γ *is quasi-Fuchsian.*

Recall that a *Fuchsian group* is a discrete torsion-free subgroup of $\mathrm{PSL}_2(\mathbb{R}) \approx \mathrm{Isom}^+(\mathbb{H}^2)$[16]. Let us just mention an important characterization of quasi-Fuchsian groups (giving them their name): a Kleinian group Γ is quasi-Fuchsian if and only if it is conjugated to a Fuchsian group by a quasiconformal[17]homeomorphism of $\mathbb{C}\mathbf{P}^1$.

[13]The limit set $\Lambda(\Gamma)$ is more classically defined as the closure in $\mathbb{C}\mathbf{P}^1 = \partial_\infty \mathbb{H}^3$ of the orbit of any point $x \in \mathbb{H}^3$ under the action of Γ.

[14] In fact better, it comes with a *complex projective structure*. I refer to [11] for an overview of complex projective structures.

[15]*i.e.* a topological circle.

[16]This definition may be relaxed to include Kleinian groups that are conjugated to subgroups of $\mathrm{PSL}_2(\mathbb{R})$ inside $\mathrm{PSL}_2(\mathbb{C})$.

[17]cf. e.g. [1] for background on quasiconformal mappings.

Simultaneous uniformization

As mentioned above, when Γ is a quasi-Fuchsian group the domain of discontinuity consists of two connected components Ω^+ and Ω^-. The Riemann surfaces $X^+ := \Omega^+/\Gamma$ and $X^- := \Omega^-/\Gamma$ are the two components of the conformal ideal boundary of the quasi-Fuchsian 3-manifold $M = \mathbb{H}^3/\Gamma$. In particular X^+ and X^- are smoothly diffeomorphic, but not necessarily conformally. As a matter of fact, Bers' beautiful simultaneous uniformization theorem claims that this pair of conformal structures can be anything:

Theorem 2.4 (Bers [5])**.** *Let S be a closed connected oriented surface of genus $g \geqslant 2$. Let X^+ and X^- be complex structures on S and $\overline{S}$[18] respectively. Then there exists a quasi-Fuchsian group Γ such that X^+ and X^- are conformally isomorphic to Ω^+/Γ and Ω^-/Γ respectively. Γ is unique up to conjugation in* $\mathrm{PSL}_2(\mathbb{C})$.

This theorem was generalized by the work of authors including Ahlfors, Bers, Kra, Marden, Maskit, Sullivan and Thurson, relating the convex cocompact hyperbolic structures on a hyperbolizable 3-manifold and the conformal structures on its ideal boundary.

Character variety and deformation space of quasi-Fuchsian structures

From now on S will denote a connected smooth closed oriented surface of genus $g \geqslant 2$.

Let $G = \mathrm{PSL}_2(\mathbb{C})$ in what follows. The *representation variety* $\mathrm{Hom}(\pi_1 S, G)$ is the set of all group homomorphisms (representations) $\rho : \pi_1 S \to G$. It has the structure of a complex affine algebraic set on which G acts algebraically by conjugation. The topological quotient is rather pathological, but the algebraic quotient (in the sense of invariant theory)[19] $\mathcal{X}(S, G) := \mathrm{Hom}(\pi_1 S, G)/\!/G$ is an affine variety, called the *character variety*. However G does act freely and properly on the subset $\mathrm{Hom}(\pi_1 S, G)^s$ of irreducible[20] ("stable") representations so that the quotient $\mathrm{Hom}(\pi_1 S, G)^s/G$ is a complex manifold, and it embeds (as a Zariski-dense open subset) in the smooth locus of $\mathcal{X}(S, G)$. More generally, the points of $\mathcal{X}(S, G)$ are in bijective correspondence with the conjugacy classes of *reductive* representations (see definition 3.1 and proposition 3.2). All representations we will be considering in these notes are irreducible or at least reductive, for this reason we need not be too concerned with the precise definition of $\mathcal{X}(S, G)$.

Next we define quasi-Fuchsian representations:

Definition 2.5. A representation $\rho : \pi_1 S \to \mathrm{PSL}_2(\mathbb{C})$ is called quasi-Fuchsian if it is faithful and its image $\Gamma < \mathrm{PSL}_2(\mathbb{C})$ is a quasi-Fuchsian group.

and the deformation space of quasi-Fuchsian structures:

Definition 2.6. The deformation space of quasi-Fuchsian structures $\mathcal{QF}(S)$ is the subset of the character variety $\mathcal{X}(S, \mathrm{PSL}_2(\mathbb{C}))$ comprising the conjugacy classes of quasi-Fuchsian representations.

[18] $\overline{S}$ denotes S with reversed orientation.

[19] I refer to [17] and [31] for more explanations on the algebraic structure of the character variety.

[20] A representation $\rho : \pi \to \mathrm{(P)SL}_2(\mathbb{C})$ is called *irreducible* (or *stable*) if it fixes no point in $\mathbb{C}\mathbf{P}^1$.

The deformation space of Fuchsian structures $\mathcal{F}(S)$ is the subset of the character variety comprising the conjugacy classes of Fuchsian representations[21], Goldman showed that it is in fact a connected component of the $\mathrm{PSL}_2(\mathbb{R})$-character variety $\mathcal{X}(S, \mathrm{PSL}_2(\mathbb{R}))$.

Proposition 2.7. *$\mathcal{QF}(S)$ is an open neighborhood $\mathcal{F}(S)$ in $\mathcal{X}(S, \mathrm{PSL}_2(\mathbb{C}))$.*

Of course $\mathcal{F}(S) \subset \mathcal{QF}(S)$, it is far less obvious that $\mathcal{QF}(S)$ is an open set in $\mathcal{X}(S, \mathrm{PSL}_2(\mathbb{C}))$. The work of Ahlfors, Bers, Kra, Marden, Maskit, Sullivan and Thurson show that the closure of $\mathcal{QF}(S)$ in $\mathcal{X}(S, \mathrm{PSL}_2(\mathbb{C}))$ is precisely the set consisting of the conjugacy classes of discrete and faithful representations, and that this set is the closure of its interior.

Let us recall here that the Teichmüller space of S is the deformation space of complex structures on S: $\mathcal{T}(S) = \{\text{complex structures on } S\}/\mathrm{Diff}_0^+(S)$, it is a complex manifold of dimension $3g - 3$. The celebrated uniformization theorem provides a real-analytic bijection $\mathcal{T}(S) \to \mathcal{F}(S)$ (associating to a conformal structure on S the unique hyperbolic structure in the conformal class).

Looking back at simultaneous uniformization, Bers' theorem 2.4 provides a bijective map $\beta : \mathcal{T}(S) \times \mathcal{T}(\overline{S}) \to \mathcal{QF}(S)$. Bers shows in fact that

Theorem 2.8. *The simultaneous uniformization map $\beta : \mathcal{T}(S) \times \mathcal{T}(\overline{S}) \to \mathcal{QF}(S)$ is a biholomorphism.*

It should be clear that the restriction of β to the diagonal is the bijection $\mathcal{T}(S) \to \mathcal{F}(S)$ given by the uniformization theorem.

Definition 2.9. For a fixed $X^+ \in \mathcal{T}(S)$ (resp. $X^- \in \mathcal{T}(\overline{S})$), the image by β of $\{X^+\} \times \mathcal{T}(\overline{S})$ (resp. $\mathcal{T}(S) \times \{X^-\}$) in $\mathcal{QF}(S) \subset \mathcal{X}(S, \mathrm{PSL}_2(\mathbb{C}))$ is called a vertical (resp. horizontal) *Bers slice*.

An important fact about Bers slices is that the so-called Schwarzian parametrization identifies the Bers slice associated to $X \in \mathcal{T}(S)$ to an open ball in the space of holomorphic quadratic differentials on X. This gives in particular a holomorphic embedding of Teichmüller space in a complex vector space of the same dimension, called the Bers embedding (see e.g. [11] for details).

Thus vertical and horizontal Bers slices are two transverse foliations of $\mathcal{QF}(S)$ by complex submanifolds which are copies of Teichmüller space. In fact they are complex Lagrangian foliations with respect to the complex symplectic structure of the character variety[22]. Andy and I discuss this bi-Lagrangian structure in an upcoming article [30].

2.2 Minimal surfaces in quasi-Fuchsian 3-manifolds

Let $M = \mathbb{H}^3/\Gamma$ be a quasi-Fuchsian 3-manifold. Let us first look at the simple situation when M is Fuchsian, *i.e.* Γ is a Fuchsian group. In that case the limit set $\Lambda \subset \mathbb{C}\mathbf{P}^1$ is a Γ-invariant circle, and its convex hull in $\mathbb{H}^3$ is a Γ-invariant totally geodesic plane H (which is a copy of $\mathbb{H}^2$ in $\mathbb{H}^3$). In the quotient $M = \mathbb{H}^3/\Gamma$, the convex core H/Γ is reduced to a totally geodesic embedded surface, in

[21] It is naturally identified with the deformation space of hyperbolic structures on S, sometimes called the Fricke space of S.

[22] By a general construction of Goldman [16] following Atiyah-Bott [4], the character variety of a closed surface always enjoys a natural symplectic structure.

particular it is a minimal surface. Since any minimal surface in M must be contained in its convex core (see below), M contains no other minimal surface.

Let $M = \mathbb{H}^3/\Gamma$ now be any quasi-Fuchsian 3-manifold. It is natural to ask whether the existence and unicity of a minimal surface in M still holds. By a convexity argument using the maximum principle (see [2]), any immersed minimal surface in M must be contained in its convex core. The work of Schoen-Yau [40], Sacks-Uhlenbeck [38], Freedman-Hass-Scott [15] and Meeks-Yau [34, 35] shows that such a minimal surface always exists, and that it can be chosen incompressible[23], embedded and area-minimizing[24]. This existence result is also a consequence of Michael Anderson's important theorem solving the asymptotic Plateau problem for submanifolds of any dimension in hyperbolic n-space:

Theorem 2.10 (Anderson [2])**.** *Let $\Gamma \subset \partial_\infty \mathbb{H}^n$ be an embedded closed submanifold in the boundary at infinity of $\mathbb{H}^n$. Then there exists a complete, absolutely area-minimizing locally integral current Σ in $\mathbb{H}^n$ asymptotic to Γ at infinity.*

When Γ is a hypersurface in $\partial_\infty \mathbb{H}^n$ and $n < 7$, regularity results of geometric measure theory imply that Σ is a smoothly embedded hypersurface in $\mathbb{H}^n$. I refer to [7] for details.

Thus any quasi-Fuchsian 3-manifold $M = \mathbb{H}^3/\Gamma$ always contains at least one minimal surface. However unicity does not hold in general. Anderson [3] showed that M contains at most a finite number of incompressible[25] stable[26] minimal surfaces, but Wang [51] and Huang-Wang [22] showed that this number can be greater than one and in fact arbitrarily large. Nonetheless, Uhlenbeck [50] showed that unicity does hold for quasi-Fuchsian structures in a neighborhood of Fuchsian structures, called *almost-Fuchsian structures*.

2.3 Almost-Fuchsian structures

Let S still denote a connected closed oriented surface of genus $g \geqslant 2$.

Recall that by the "fundamental theorem of surface theory" 1.19.a, any minimal hyperbolic germ $(g, \mathrm{I\!I})$ on S gives rise to a unique minimal immersion of S into the germ a hyperbolic thickening M. In the foundational paper [50], Uhlenbeck wrote down the explicit expression of the hyperbolic metric on M under the normal exponential map of the minimal surface. She showed that given the appropriate control of the principal curvatures of the minimal surface, namely that they are everywhere in $(-1, 1)$[27], the minimal surface Σ is smoothly embedded and the hyperbolic metric extends to a complete hyperbolic metric on $M \approx S \times \mathbb{R}$. Moreover, the hyperbolic manifold M is quasi-Fuchsian and contains no other minimal surface. Let us summarize these results:

[23] A map $f : S \to M$ is called *incompressible* if the induced map on fundamental groups $f_* : \pi_1 S \to \pi_1 M$ is injective.

[24] An immersion $f : S \to M$ is called *area-minimizing* if its area is less than any other immersion in the same homotopy class. An area-minimizing immersion is in particular a minimal immersion.

[25] He also shows that if the incompressibility condition is dropped, then M may contain infinitely many minimal surfaces, which must have infinite genera.

[26] A minimal surface is called *stable* if the second variation of the area functional on compactly supported normal deformations is always non-negative.

[27] An immersed surface in a hyperbolic 3-manifold with principal curvatures everywhere in $(-1, 1)$ is sometimes called *strictly horospherically convex*. Horospheric convexity that the surface remains on the concave side of the tangent horosphere at each point.

Theorem 2.11 (Uhlenbeck [50])**.** *Let $(g, \mathrm{I\!I})$ be a minimal hyperbolic germ. Assume that $\| \mathrm{I\!I} \|_g^2 < 2$ everywhere on S*[28]*. Then*

(i) There exists a unique immersion f of S into a complete *hyperbolic 3-manifold M (up to isometry) such that $\mathrm{I}(f) = \mathrm{I}$ and $\mathrm{I\!I}(f) = \mathrm{I\!I}$.*

(ii) f is a smooth incompressible two-sided embedding.

(iii) Let Σ denote the minimal surface $f(S) \subset M$ and $T^\perp\Sigma \subset TM$ denote the normal bundle to Σ. Then $\exp : T^\perp\Sigma \approx S \times \mathbb{R} \to M$ is a global diffeomorphism. The pull-back of the hyperbolic metric on $S \times \mathbb{R}$ is expressed as

$$dt^2 + g(\cosh(t)\mathbf{1}(\cdot) + \sinh(t)B(\cdot), \cosh(t)\mathbf{1}(\cdot) + \sinh(t)B(\cdot)) \tag{2.1}$$

where $\mathbf{1}$ denotes the identity operator and B is the shape operator (see definition 1.10).

(iv) M is a quasi-Fuchsian 3-manifold.

(v) Σ is the only closed minimal surface in M.

This motivates the following definitions:

Definition 2.12. The term *almost-Fuchsian* may refer to:

- A quasi-Fuchsian 3-manifold is called almost-Fuchsian if it contains a minimal surface with principal curvatures everywhere in $(-1, 1)$.
- A minimal hyperbolic germ $(g, \mathrm{I\!I})$ is called almost-Fuchsian if $\| \mathrm{I\!I} \|_g^2 < 2$ everywhere.
- A Kleinian group Γ is called almost-Fuchsian if the hyperbolic 3-manifold $\mathbb{H}^3/\Gamma$ is almost-Fuchsian.
- A representation $\rho : \pi_1 S \to \mathrm{PSL}_2(\mathbb{C})$ is called almost-Fuchsian if it is faithful and its image $\rho(\pi_1 S) < \mathrm{PSL}_2(\mathbb{C})$ is almost-Fuchsian.

We proceeed to define the deformation space of almost-Fuchsian structures:

Definition 2.13. The deformation space of almost-Fuchsian structures $\mathcal{AF}(S)$ is the subset of the character variety $\mathcal{X}(S, \mathrm{PSL}_2(\mathbb{C}))$ comprising the conjugacy classes of almost-Fuchsian representations $\rho : \pi_1 S \to PSL_2(\mathbb{C})$.

It is not too hard to believe that

Proposition 2.14. *$\mathcal{AF}(S)$ is a neighborhood of $\mathcal{F}(S)$ in $\mathcal{QF}(S)$.*

Allow me to quote [39] here: *The structure of almost-Fuchsian manifolds has been studied considerably by a number of authors. In particular, the invariants arising from quasi-conformal Kleinian group theory (e.g. Hausdorff dimension of limit sets, distance between conformal boundary components) are controlled by the principal curvatures of the unique minimal surface.* Relating the geometry of the minimal surface to the geometry of the boundary at infinty is indeed a key problem whose study generates interesting results[29]. Let us mention the following useful fact in that respect:

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[28] Equivalently, the eigenvalues of the g-self-adjoint operator associated to $\mathrm{I\!I}$ (*i.e.* the potential principal curvatures) λ_1 and $\lambda_2 = -\lambda_1$ are in $(-1, 1)$ everywhere on S.

[29] My paper [29] follows that philosophy, using an *ad hoc* notion of renormalized volume (following Krasnov-Schlenker [26, 27]) to relate the symplectic structure of the moduli space of almost-Fuchsian structures "seen from the minimal surface" to its symplectic structure "seen from infinity".

Proposition 2.15. *Let M be the almost Fuchsian 3-manifold associated to an almost-Fuchsian minimal hyperbolic germ (g, II). Denote by X the Riemann surface structure on the minimal surface given by the conformal class of g. Let z be a local complex coordinate on X, write the metric as $g = e^{2u}|dz|^2$ and the second fundamental form as $\mathrm{II} = \frac{1}{2}(\alpha(z)dz^2 + \overline{\alpha(z)}d\bar{z}^2)$, where $\alpha = \alpha(z)dz^2$ is a holomorphic quadratic differential on X (see 1.20). Let $\mu = \mu(z)\frac{d\bar{z}}{dz}$ be the Beltrami differential given by $\mu = g^{-1}\bar{\alpha} = e^{-2u}\overline{\alpha(z)}\frac{d\bar{z}}{dz}$. Then the metrics*

$$|dz \pm \mu(z)d\bar{z}|^2 \tag{2.2}$$

are conformal metrics on the conformal ideal boundary components $\partial^+_\infty M$ and $\partial^-_\infty M$ respectively.

Let us outline a sketch of proof.

Proof. It is straightforward to check that the metric (2.1) in the complex coordinates z is expressed as

$$dt^2 + e^{2u}|\cosh(t)dz + \sinh(t)\, e^{-2u}\overline{\alpha(z)}\, d\bar{z}|^2 \tag{2.3}$$

as Fock observed in [13]. The metric $|\cosh(t)dz + \sinh(t)\, e^{-2u}\overline{\alpha(z)}\, d\bar{z}|^2$ on S is the expression of the induced metric g_t on the embedded surface $S \times \{t\} \subset S \times \mathbb{R} \approx M$ that is the image of the minimal surface $\Sigma \approx S \times \{0\}$ under the time t-normal exponential map. One can show that for any horospherically convex incompressible embedded surface $\Sigma \subset M$, the conformal structure on the time t-normal exponential image of Σ must converge to the conformal structure of the boundary at infinity as $t \to +\infty$. Here g_t is asymptotic to $e^{2u}\frac{e^t}{2}\,|dz \pm \mu(z)d\bar{z}|^2$ as $t \to \pm\infty$, the result follows. □

2.4 Taubes moduli space

In his paper [47], Taubes introduced minimal hyperbolic germs (see definition 1.18) and their deformation space. Diffeomorphisms of the surface S naturally act on minimal hyperbolic germs by pull-back, as always the deformation space is the quotient by the subgroup of homotopically trivial diffeomorphisms:

Definition 2.16. Let S be a connected closed oriented surface of genus $g \geqslant 2$. The *deformation space of minimal hyperbolic germs* or *Taubes moduli space* is

$$\mathcal{H} := \{(g, \mathrm{II}) \text{ minimal hyperbolic germ on } S\}/\mathrm{Diff}_0(S)\,. \tag{2.4}$$

Taubes shows that this deformation space has a natural structure of a manifold of the expected dimension (with no singularities), and that it comes with a natural symplectic structure $\omega_{\mathcal{H}}$ and $U(1)$-action[30]:

Theorem 2.17. *The Taubes moduli space $\mathcal{H}$ is a smooth manifold of dimension $12g - 12$ equipped with a real symplectic structure $\omega_{\mathcal{H}}$ and a smooth $U(1)$-action.*

[30] $U(1)$ denotes the group of unit complex numbers. A $U(1)$-action is often called a "S^1-action".

The symplectic structure $\omega_{\mathcal{H}}$ that can be obtained by symplectic reduction from the canonical symplectic structure on the cotangent bundle of the (infinite-dimensional) space of Riemannian metrics on S. We refer to Taubes' paper [47] for details, but note that $\omega_{\mathcal{H}} = \mathrm{Re}(\Psi^* \omega_c)$ can be taken for a definition of a $\omega_{\mathcal{H}}$, see theorem 2.18 (ii) below. The $U(1)$-action can be described by rotating $\mathrm{I\!I}$ in a g-orthonormal frame, one can also consider the $U(1)$-equivariance of Ψ as its definition, again see theorem 2.18 (ii) below.

Taubes defines and studies two important "canonical" maps $\Phi : \mathcal{H} \to \mathcal{X}(S, \mathrm{PSL}_2(\mathbb{C}))$ and $\Psi : \mathcal{H} \to T^*\mathcal{T}(S)$, let us introduce these maps. First the map Φ: recall that by theorem 1.19.b, given a minimal hyperbolic germ $(g, \mathrm{I\!I})$ there exists a unique minimal immersion $\tilde{f} : \tilde{S} \to \mathbb{H}^3$ such that $\mathrm{I}(f)$ and $\mathrm{I\!I}(f)$ are the lifts of g and $\mathrm{I\!I}$ to $\tilde{S}$ and f is ρ-equivariant for some representation $\rho : \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$. The representation ρ is unique up to conjugation, so the assignement $(g, \mathrm{I\!I}) \mapsto \rho$ gives a well-defined map $\Phi : \mathcal{H} \to \mathcal{X}(S, \mathrm{PSL}_2(\mathbb{C}))$.

Now the map Ψ: Given a minimal hyperbolic germ $(g, \mathrm{I\!I})$, let X denote the complex structure on S given by the conformal class of g. Recall that $\mathrm{I\!I}$ is the real part of a holomorphic quadratic differential $\alpha = 2\,\mathrm{I\!I}^{(2,0)}$ on the Riemann surface X (see proposition 1.20). The space of holomorphic quadratic differentials on X is naturally identified to the complex cotangent space at X to Teichmüller space $\mathcal{T}(S)$[31]. The assignement $(g, \mathrm{I\!I}) \to (X, \alpha)$ thus defines a map $\Psi : \mathcal{H} \to T^*\mathcal{T}(S)$ after taking the quotient by the action of $\mathrm{Diff}_0(S)$.

The following theorem summarizes Taubes' results:

Theorem 2.18. *Let $\Phi : \mathcal{H} \to \mathcal{X}(S, \mathrm{PSL}_2(\mathbb{C}))$ and $\Psi : \mathcal{H} \to T^*\mathcal{T}(S)$ be the maps as above.*

(i) Φ is a smooth map and a real symplectomorphism with respect to the symplectic structures $\omega_{\mathcal{H}}$ on $\mathcal{H}$ and $\mathrm{Im}(\omega_G)$ on $\mathcal{X}(S, \mathrm{PSL}_2(\mathbb{C}))$.

(ii) Ψ is a smooth map, it is equivariant for the $U(1)$-action[32] a real symplectomorphism with respect to the symplectic structures $\omega_{\mathcal{H}}$ on $\mathcal{H}$ and $\mathrm{Re}(\omega_c)$ on $T^*\mathcal{T}(S)$.

(iii) Φ and Ψ have the same set of critical points in $\mathcal{H}$. At such points the kernels of their derivatives are in direct sum. It follows that $\Phi \oplus \Psi$ is an Lagrangian immersion of $\mathcal{H}$ into $\mathcal{X}(S, \mathrm{PSL}_2(\mathbb{C})) \oplus T^*\mathcal{T}(S)$ for the appropriate symplectic structure.

Let us clarify the symplectic structures a bit. The symplectic structure ω_G denotes the complex symplectic structure of the character variety $\mathcal{X}(S, \mathrm{PSL}_2(\mathbb{C}))$. Indeed, a general construction of Goldman [16] following Atiyah-Bott [4] shows that the character variety $\mathcal{X}(S, G)$ of a closed surface always enjoys a natural symplectic structure (provided G is a reductive Lie group). When G is a complex Lie group, ω_G is a complex symplectic form. In [16], Goldman also shows that when $G = \mathrm{PSL}_2(\mathbb{R})$ the symplectic struture ω_G restricted to the Teichmüller component $\mathcal{F}(S)$ coincides with the Weil-Petersson Kähler form[33] on Teichmüller space $\mathcal{T}(S)$ under the identification $\mathcal{T}(S) \approx \mathcal{F}(S)$ given by the uniformization theorem. Note that this fact can be recovered from Taubes' result. The symplectic structure ω_c denotes the complex symplectic structure of the holomorphic cotangent

[31] This is a standard fact in Teichmüller theory. It is classically explained in terms of quasiconformal deformations of complex structures, but can also and more intrinsically be derived from Kodaira-Spencer deformation theory.

[32] The $U(1)$-action on the holomorphic cotangent bundle $T^*\mathcal{T}(S)$ is just the action by complex multiplication in the fibers.

[33] The Teichmüller space $\mathcal{T}(S)$ enjoys a natural Kähler structure classically described as the Weil-Petersson product of holomorphic quadratic differentials, see e.g. [52] for an overview.

bundle $T^*\mathcal{T}(S)$, which has a canonical complex symplectic structure like any holomorphic cotangent bundle.

Taubes also shows that the maps Φ and Ψ have no critical points on the open subspace corresponding to almost-Fuchsian germs $\mathcal{H}_{\mathcal{AF}} \subset \mathcal{H}$. The restriction of Φ to $\mathcal{H}_{\mathcal{AF}}$ is a diffeomorphism of $\mathcal{H}_{\mathcal{AF}}$ onto $\mathcal{AF}(S) \subset \mathcal{X}(S, \mathrm{PSL}_2(\mathbb{C}))$, which is simply given by assigning to an almost-Fuchsian germ the corresponding almost-Fuchsian representation. However Φ is not injective on the whole moduli space $\mathcal{H}$, this is reflected by the fact that a hyperbolic 3-manifold that is not almost-Fuchsian can contain several minimal surfaces[34]. The restriction of Ψ to $\mathcal{H}_{\mathcal{AF}}$ is a diffeomorphism of $\mathcal{H}_{\mathcal{AF}}$ onto a (somewhat mysterious) open neighborhood of the zero section in $T^*\mathcal{T}(S)$. Note that the space of *Fuchsian germs* $\mathcal{H}_{\mathcal{F}} \subset \mathcal{H}$ corresponding to germs of the form $(g, 0)$ where g is a hyperbolic metric on S is naturally identified to the Fricke deformation space of hyperbolic structures on S, identified itself to the deformation space of Fuchsian structures $\mathcal{F}(S)$ via holonomy. The restriction of Φ to $\mathcal{H}_{\mathcal{F}}$ is precisely that identification. On the other hand the restriction of Ψ to $\mathcal{H}_{\mathcal{F}}$ sends $\mathcal{H}_{\mathcal{F}}$ diffeomorphically to the zero section of $T^*\mathcal{T}(S)$, it is precisely the bijection between hyperbolic metrics and complex structures on S given by the uniformization theorem.

3 Higgs bundles and minimal surfaces

Blabla about the theory of Higgs bundles

Throughout this section S still denote a connected closed oriented surface of genus $g \geqslant 2$.

3.1 $\mathrm{SL}_2(\mathbb{C})$-Higgs bundles and the non-abelian Hodge correspondence

Reductive representations and Corlette's theorem

Let $G = (\mathrm{P})\mathrm{SL}_2(\mathbb{C})$ and let $\mathcal{X}(S) = \mathcal{X}(S, G)$ denote the $(\mathrm{P})\mathrm{SL}_2(\mathbb{C})$-character variety. In this section we will not be concerned with the distinction between the theories for $G = \mathrm{PSL}_2(\mathbb{C})$ and $G = \mathrm{SL}_2(\mathbb{C})$[35].

Definition 3.1. A representation $\rho : \pi_1 S \to (\mathrm{P})\mathrm{SL}_2(\mathbb{C})$ is called *reductive* (or *semisimple*, or *polystable*) if the action of $\rho(\pi_1 S)$ on $\mathbb{C}\mathbf{P}^1$ does not fix exactly one point[36].

It is worth mentioning that

[34] Actually I do not know for sure that Φ is not injective, but it seems to me that it is a consequence of Biao Wang's result [51] that a quasi-Fuchsian manifold can contain several incompressible non-isotopic closed minimal surfaces of the same genus. I am also unsure as to whether Ψ is injective on the whole moduli space $\mathcal{H}$.

[35] especially since all the representations $\rho : \pi_S \to \mathrm{PSL}_2(\mathbb{C})$ being considered in these notes are liftable to $\mathrm{SL}_2(\mathbb{C})$. The advantage of taking $G = \mathrm{SL}_2(\mathbb{C})$ over $G = \mathrm{PSL}_2(\mathbb{C})$ is that the theory of Higgs bundles in the linear case can be described in terms of vector bundles instead of principal bundles. For the interested reader, let us mention that the $\mathrm{PSL}_2(\mathbb{C})$-character variety splits into two irreducible components reflecting the liftability of corresponding representations, and the moduli space of $\mathrm{PSL}_2(\mathbb{C})$-Higgs bundles has an analogous decomposition.

[36] This is of course the adaptation for $G = (\mathrm{P})\mathrm{SL}_2(\mathbb{C})$ of a more general notion of reductivity for any reductive Lie group G, which is the following: a representation $\rho : \Gamma \to G$ is called reductive if the Zariski closure of $\rho(\Gamma)$ in G is a reductive subgroup. Equivalently, ρ acts completely reducibly on the Lie algebra of G via the adjoint representation.

Proposition 3.2. *Let* $\mathrm{Hom}^{ss}(\pi_1 S, G) \subset \mathrm{Hom}(\pi_1 S, G)$ *denote the space of reductive representations. There is a natural map* $\mathrm{Hom}(\pi_1 S, G)/G \to \mathrm{Hom}(\pi_1 S, G)//G =: \mathcal{X}(S)$ *which restricts to a bijection* $\mathrm{Hom}^{ss}(\pi_1 S, G)/G \to \mathcal{X}(S)$.

The following foundational theorem is due to Donaldson [9]. It was generalized by Corlette [6] for any reductive Lie group G (and further by Labourie [28]).

Theorem 3.3. *Let* X *be a complex structure on* S *and let* $\rho : \pi_1(S) \to PSL_2(\mathbb{C})$ *be a representation. There exists a* ρ*-equivariant harmonic map* $f : \tilde{X} \to \mathbb{H}^3$ *if and only if* ρ *is reductive. A such map* f *is unique up to post-composition with an element of* $\mathrm{PSL}_2(\mathbb{C})$ *which centralizes* $\rho(\pi_1 S)$[37].

$SL_2(\mathbb{C})$-Higgs bundles

Let $G = \mathrm{SL}_2(\mathbb{C})$ here. Let X be a complex structure on S (*i.e.* X is a Riemann surface).

Definition 3.4. A $\mathrm{SL}_2(\mathbb{C})$-*Higgs bundle* on X is a pair $(\mathcal{E}, \varphi)$ where

- $\mathcal{E}$ is a rank 2 holomorphic vector bundle on X with trivial determinant[38].
- φ is a holomorphic $(1,0)$-form on X with values in the bundle of traceless endomorphisms of $\mathcal{E}$. In other words $\varphi \in H^0(K_X \otimes \mathrm{End}(E))$ with $\mathrm{tr}\,\varphi = 0$, where $K_X = T^*X$ is the canonical bundle on X.

In order to define the moduli space we restrict our attention to polystable Higgs bundles:

Definition 3.5. A $\mathrm{SL}_2(\mathbb{C})$-Higgs bundle $(\mathcal{E}, \varphi)$ on X is *polystable* if either:

(i) every holomorphic φ-invariant subbundle L has negative degree, in this case the Higgs bundle is called *stable*,

or

(ii) $\mathcal{E}$ splits (holomorphically) as a sum of two φ-invariant line subbundles of degree zero.

We may now define the moduli space of Higgs bundles up to gauge equivalence:

Definition 3.6. Let X be a complex structure on S. The *moduli space of polystable* $\mathrm{SL}(2,\mathbb{C})$-*Higgs bundles on* X (or *Dolbeault moduli space*) is

$$\mathcal{M}(X) := \{(\mathcal{E}, \varphi) \text{ polystable } \mathrm{SL}(2,\mathbb{C}))\text{-Higgs bundle on } X\}/\sim \tag{3.1}$$

where two Higgs bundle $(\mathcal{E}, \varphi)$ and $(\mathcal{E}', \varphi')$ are equivalent if there exists an isomorphism $t : \mathcal{E} \to \mathcal{E}'$ such that $t^*\varphi' = \varphi$.

Hitchin gave an analytic construction of $\mathcal{M}(X)$ in [18] and [19], showing in particular that

Theorem 3.7. $\mathcal{M}(X)$ *is an irreducible quasiprojective algebraic variety of complex dimension* $6g-6$. *Also:*

[37] Note that the centralizer of $\rho(\pi_1 S)$ is trivial as soon as ρ is irreducible, so the equivariant harmonic map f is unique for at least irreducible representations.

[38] *i.e.* $\Lambda^2\mathcal{E} \approx \mathcal{O}$ where $\mathcal{O}$ is the trivial line bundle on X. Note that a vector bundle with trivial determinant has zero degree.

(i) The set of stable points defines a dense smooth open subvariety.
(ii) $\mathcal{M}(X)$ enjoys a natural hyperkähler structure.

Let us also define the Hitchin fibration. Given a Higgs field $(\mathcal{E}, \varphi)$, the trace $\mathrm{tr}(\varphi^2) = -2 \det \varphi$ is a section of K_X^2, *i.e.* a holomorphic quadratic differential on X.

Definition 3.8. The *Hitchin fibration map* $F_X : \mathcal{M}(X) \to H^0(X, K_X^2)$ is the map induced by $(\mathcal{E}, \varphi) \mapsto \mathrm{tr}(\varphi^2)$. The *nilpotent cone*[39] is $F_X^{-1}(0) \subset \mathcal{M}(X)$.

Non-abelian Hodge correspondence

The celebrated *non-abelian Hodge correspondence* is the natural correspondence between reductive representations and polystable Higgs bundles discovered by Hitchin and Simpson, inducing an bijective correspondence between moduli spaces $\Theta_X : \mathcal{X}(S) \leftrightarrow \mathcal{M}(X)$. The key result to this correspondence to go from representations to Higgs bundles is the theorem of Donaldson and Corlette previously mentioned (theorem 3.3). The other direction, from Higgs bundles to representations, relies on the parallel theorem of Hitchin and Simpson showing the existence of harmonic metrics solving the self-duality equations for polystable Higgs bundles ([18], [42], [43, 44]).

Let us describe the non-abelian Hodge correspondence in the direction $\mathcal{X}(S) \to \mathcal{M}(X)$ for $G = (\mathrm{P})\mathrm{SL}_2(\mathbb{C})$. Então, Let $\rho : \pi \to SL(2, \mathbb{C})$ be a reductive representation. Let $E_\rho \to S$ be the flat rank 2 complex vector bundle associated to the flat $\pi_1(S)$-bundle $\tilde{S} \to S$ by the action of $\pi_1 S$ on $\mathbb{C}^2$ via ρ[40]. It remains to construct a holomorphic structure $\bar{\partial}_E$ on E_ρ and a Higgs field $\varphi \in \Omega^{(1,0)}(E_\rho)$.

By Donaldson's theorem, there exists an equivariant harmonic map $f_\rho : \mathbb{H}^2 \approx \tilde{X} \to \mathbb{H}^3$. Since $\mathbb{H}^3 \approx G/K$ where $G = \mathrm{SL}_2(\mathbb{C})$ and $K = SU(2)$, this map gives a reduction of the structure group of E_ρ to $SU(2)$, in other words it defines a Hermitian metric h in E_ρ (and by definition, this metric is harmonic). We define the holomorphic structure $\bar{\partial}_E$ as the $(0, 1)$-part of the unitary part A of the flat connection on E_ρ.

Consider then the derivative $df_\rho \in \Omega^1(\tilde{S}, {f_\rho}^*(T\mathbb{H}^3))$. The bundle ${f_\rho}^*(T\mathbb{H}^3) \to S$ can be identified with the $Ad \circ \rho$-bundle with typical fiber $\mathfrak{m} \approx T_{[e]}G/K$. Here we have written $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C}) = \mathfrak{k} \oplus \mathfrak{m}$, where $\mathfrak{k} = \mathfrak{su}(2)$ and $\mathfrak{m} = i\mathfrak{k}$ (traceless Hermitian matrices). Also, df_ρ is $\pi_1(S)$-invariant so it descends to a one-form on S. Under these identifications, let $\psi = df_\rho \in \Omega^1(\mathfrak{m})$ and $\varphi = \psi^{1,0} \in \Omega^{1,0}(\mathfrak{g})$. Note that the $\mathrm{Ad} \circ \rho$-bundle $\mathfrak{g} \to S$ is precisely the bundle $\mathrm{Ad}(E_\rho)$, in other words φ is a one-form with values in traceless endomorphisms of E_ρ as required.

It remains to show that φ is holomorphic as a one-form with values in the endomorphisms of the holomorphic vector bundle $\mathcal{E} = (E, \bar{\partial}_E)$, in other words that $\bar{\partial}_E \varphi = 0$. But it is a standard computation that the harmonicity of f_ρ is equivalent to Hitchin's *self-duality equations*:

$$F(A) + [\varphi, \varphi^{*_h}] = 0 \tag{3.2}$$
$$\bar{\partial}_E \varphi = 0 \tag{3.3}$$

where A denotes the unitary part with respect to the harmonic metric h of the flat connection on E_ρ and $F(A)$ denotes the curvature of A. Thus $(\mathcal{E}, \varphi)$ is a Higgs bundle as we wanted.

[39] The reason for this terminology is that if $(\mathcal{E}, \varphi)$ is in the nilpotent cone, then $\mathrm{tr}(\varphi) = \mathrm{tr}(\varphi^2) = 0$, so φ takes values in nilpotent endomorphisms.

[40] Concretely, E_ρ may be described as the suspension $\tilde{S} \times \mathbb{C}^2 / \pi_1(S)$, where $\gamma \cdot (x, v) = (\gamma \cdot x, \rho(\gamma) \cdot v)$.

Let us now sketch the inverse direction, going from Higgs bundles to representations. So, start with a Higgs bundle $(\mathcal{E}, \varphi)$. Let us now regard the self-duality equation (3.2) as an equation for a Hermitian metric h on $\mathcal{E}$, where A now denotes the Chern connection associated to h and the holomorphic structure on $\mathcal{E}$ (the equation (3.3) on the other hand is now automatically satisfied). The theorem of Hitchin and Simpson goes:

Theorem 3.9. *There exists a Hermitian metric h such that the self-duality equations for the Higgs bundle $(\mathcal{E}, \varphi)$ are satisfied if and only if $(\mathcal{E}, \varphi)$ is polystable. Moreover, a such metric is unique.*

It remains to say how one gets a representation $\rho : \pi_1 S \to G$ out of this. A small computation shows that h satisfying the self-duality equations for $(\mathcal{E}, \varphi)$ is equivalent to the flatness of the connection $D = A + \psi$, where ψ is the h-self-adjoint one-form with values in $\operatorname{End}\mathcal{E}$ given by $\psi = \varphi + \varphi^{*h}$. The representation $\rho : \pi_1 S \to G$ is then collected as the holonomy of the flat connection D.

3.2 Minimal surfaces and Higgs bundles

Hopf differential and nilpotent cone

Let X still denote a fixed complex structure on S. Let $\rho : \pi_1 S \to \mathrm{SL}_2(\mathbb{C})$ be a reductive representation, $f_\rho : \tilde{X} \to \mathbb{H}^3$ be the ρ-equivariant harmonic map given by Donaldson's theorem and $(\mathcal{E}, \varphi)$ the associated Higgs bundle.

Recall that the Hopf differential of f_ρ is given by $\mathrm{Hopf}(f_\rho) = (f_\rho^* h_{\mathbb{H}^3})^{(2,0)}$ (see definition 1.14), where $h_{\mathbb{H}^3}$ is the hyperbolic metric on $\mathbb{H}^3 \approx G/K$. It is a holomorphic quadratic differential on X, since f_ρ is harmonic (proposition 1.15). A small computation shows that under the identifications explained in the previous paragraph in order to process the non-abelian Hodge correspondence, the Hopf differential of f_ρ is precisely (the lift of) $\mathrm{tr}(\varphi^2)$[41]. In other words:

Proposition 3.10. *Let $(\mathcal{E}, \varphi)$ be the Higgs bundle on the Riemann surface X associated to a reductive representation ρ under the non-abelian Hodge correspondence. Recall that $f_\rho : \tilde{X} \to \mathbb{H}^3$ denotes the ρ-equivariant harmonic map given by Donaldson's theorem and $F_X : \mathcal{M}(X) \to H^0(K_X^2)$ denotes the Hitchin fibration map (see definition 3.8). Then*

$$F_X(\mathcal{E}, \varphi) = \mathrm{Hopf}(f_\rho)\,. \tag{3.4}$$

In particular, using proposition 1.15 (ii), we find that

Proposition 3.11. *With the notations of the previous proposition, a polystable Higgs bundle $(\mathcal{E}, \varphi)$ lies in the nilpotent cone $F_X^{-1}(0) \subset \mathcal{M}(X)$ if and only if $f_\rho : \tilde{S} \to \mathbb{H}^3$ is a ρ-equivariant minimal immersion.*

Conversely, it is clear that any ρ-equivariant minimal immersion $f : \tilde{S} \to \mathbb{H}^3$ for some reductive ρ produces a Higgs bundle on S equipped with the complex structure making f conformal via the

[41] In a nutshell, the computation is $\mathrm{Hopf}(f) = (h_{\mathbb{H}^3}(df, df))^{(2,0)} = h_{G/K}(\varphi, \varphi) = \mathrm{tr}\,\varphi^2$.

procedure producing the non-abelian Hodge correspondence. Recall that such maps f are precisely parametrized by Taubes' minimal hyperbolic germs... Let us encapsulate these observations in a formal setting.

Definition 3.12. The *global Dolbeault moduli space* is the fiber bundle $\mathcal{M} \xrightarrow{\pi} \mathcal{T}(S)$ whose fiber above a point $X \in \mathcal{T}(S)$ is the Dolbeault moduli space $\mathcal{M}(X)$.

We will not worry about the topology or smooth structure of $\mathcal{M}$ here. As a set, $\mathcal{M}$ is just $\coprod_{X\in\mathcal{T}(S)} \mathcal{M}(X)$. I was tempted to call $\mathcal{M}$ the "universal Dolbeault moduli space" but it is most likely a silly choice of words. Terminology aside, a case for this definition could be made by arguing that studying the variation of the structure of $\mathcal{M}(X)$ as the complex structure X varies is an important (and hard) question, in that event this definition is somewhat natural.

Let us introduce a couple more notations:

Definition 3.13. Let $\mathcal{M}$ be the global Dolbeault moduli space defined above.

(i) The global Hitchin fibration is the map $F : \mathcal{M} \to Q(S)$ whose restriction to the fiber $\mathcal{M}(X)$ is the Hitchin fibration map $F_X : \mathcal{M}(X) \to H^0(K_X^2)$. Here $Q(S)$ is the bundle over Teichmüller space whose fiber above $X \in \mathcal{T}(S)$ is $H^0(X, K_X^2) =: Q(X)$.

(ii) The global non-abelian correspondence map is the map $\Theta : \mathcal{X}(S) \times \mathcal{T}(S) \to \mathcal{M}$ such that $\Theta(\cdot, X) := \Theta_X$ is the bijection $\mathcal{X}(S) \to \mathcal{M}(X)$ given by the non-abelian Hodge correspondence for the complex structure X.

Note that F and Θ are both bundle maps (between fiber bundles over $\mathcal{T}(S)$), which is not saying much. Naturally, we shall call $F^{-1}(0)$ the *nilpotent cone of* $\mathcal{M}$, comprising the Higgs bundles on any $X \in \mathcal{T}(S)$ with nilpotent Higgs field.

Our previous observations can be summarized as follows. First let us just note that Taubes' moduli space $\mathcal{H}$ also fibers over $\mathcal{T}(S)$, via the assignement $(g, \mathrm{I\!I}) \mapsto [g]$, where $[g]$ denotes the complex structure on S given by the conformal class of g[42], let us denote $\pi_{\mathcal{H}} : \mathcal{H} \to \mathcal{T}(S)$ this bundle projection. Recall that $\Phi : \mathcal{H} \to \mathcal{X}(S)$ is the "canonical" map introduced in section 2.4. Consider the map

$$\Xi : \mathcal{H} \to \mathcal{M} \tag{3.5}$$

given by

$$\Xi : \mathcal{H} \xrightarrow{\Phi\times\pi_{\mathcal{H}}} \mathcal{X}(S) \times \mathcal{T}(S) \xrightarrow{\Theta} \mathcal{M}\,, \tag{3.6}$$

in other words Ξ is the Higgs bundle associated to a minimal hyperbolic germ under the non-abelian Hodge correspondence for the appropriate complex structure (the conformal structure on the minimal surface). Then

Proposition 3.14. *$\Xi : \mathcal{H} \to \mathcal{M}$ is a bundle map which sends $\mathcal{H}$ surjectively onto the nilpotent cone $F^{-1}(0) \subset \mathcal{M}$.*

More surprising is that this map can be computed completely explicitely, as Donaldson demonstrated in [10]. We review this construction in the following subsection.

[42]By the way, the reason why I keep identifying conformal and complex structures on S is that S is oriented.

3.3 Explicit Higgs bundles associated to minimal germs

Let $(g, \mathrm{I\!I})$ be a minimal hyperbolic germ on S. Denote by X the complex structure on S given by the conformal class of g and $\alpha = 2\,\mathrm{I\!I}^{(2,0)} \in H^0(X, K_X^2)$ the holomorphic quadratic differential such that $\mathrm{I\!I} = \mathrm{Re}(\alpha)$. Let us construct a Higgs bundle on X using the data $(g, \mathrm{I\!I})$, then argue that it must be $\Xi(g, \mathrm{I\!I})$.

Fix a choice $K_X^{\frac{1}{2}}$ of a square root of the canonical bundle K_X[43] and let $L = K_X^{-\frac{1}{2}}$. Let E be the smooth complex vector bundle $E = L \oplus L^{-1}$ (let us stress that we have not equipped E with a holomorphic structure yet, so this splitting is not holomorphic *a priori*). The metric g induces a Hermitian metric on L and L^{-1}, let a^+ and a^- denote the corresponding Chern connections[44]. With respect to the decomposition $E = L \oplus L^{-1}$, the matrix

$$A = \begin{pmatrix} a^+ & \alpha^* \\ -\alpha & a^- \end{pmatrix} \tag{3.7}$$

represents a connection on E. Here α is regarded as an element of $\Omega^{(1,0)}(\mathrm{End}(L, L^{-1}))$ $(= \Omega^0(K_X^2))$ and α^* is the dual of α with respect to the metric, it is an element of $\Omega^{0,1}(\mathrm{End}(L^{-1}, L))$ $(= \Omega^0(\overline{K_X} \otimes K_X^{-1})$[45]$)$. Note that the connection A is unitary with respect to the Hermitian metric h on E induced from L and L^{-1}. Let $\bar{\partial}_E = A^{(0,1)}$ be the holomorphic structure associated to that connection. Alternatively, this holomorphic structure on E could be defined by virtue of the following proposition:

Proposition 3.15. *The holomorphic vector bundle $\mathcal{E} = (E, \bar{\partial}_E)$ is the extension of L^{-1} by L*

$$0 \to L \to E \to L^{-1} \to 0 \tag{3.8}$$

associated to the extension class $[\alpha^*] \in H^1(\mathrm{End}(L^{-1}, L))$.

Here I would like to point out that when $(g, \mathrm{I\!I})$ is an almost-Fuchsian germ, $\pm[\alpha^*]$ is also the Dolbeault cohomology class of Beltrami differentials parametrizing the difference between the complex structure on the minimal surface and the complex structure on the conformal boundary at infinity, by proposition 2.15. I certainly want to explore this observation further.

Now we define the Higgs field φ simply by

$$\varphi = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{3.9}$$

Here "1" represents an element of $\Omega^{(1,0)}(\mathrm{End}(L^{-1}, L)) = \Omega^0(K_X \otimes L^2) \approx \mathbb{C}$.

It is straightforward to check that $\bar{\partial}_E \varphi = 0$, so the pair $(\mathcal{E} = (E, \bar{\partial}_E), \varphi)$ is a Higgs bundle on X. It remains to show:

[43] For the diligent reader: a choice of square root of the canonical bundle is equivalent to a choice of a *spin structure* on X. Different choices of spin structures will produce different representations $\pi_1 S \to SL_2(\mathbb{C})$ (thereupon different $SL_2(\mathbb{C})$-Higgs bundles), but which all project to the same representation $\rho : \pi_1 S \to \mathrm{PSL}_2(\mathbb{C})$ (and so define the same $\mathrm{PSL}_2(\mathbb{C})$-Higgs bundle, although we have not defined such an object in these notes).

[44] Note: $a^- = -a^+$ in the sense that their connection 1-forms are opposite in any complex local coordinate.

[45] in other words α^* is a smooth Beltrami differential on X.

Theorem 3.16. *$(\mathcal{E}, \varphi)$ is the Higgs bundle on X corresponding to the representation $\rho : \pi_1 S \to \mathrm{SL}_2(\mathbb{C})$ associated to the minimal hyperbolic germ $(g, \mathrm{I\!I})$. More precisely: $(\mathcal{E}, \varphi) = \Xi(g, \mathrm{I\!I})$.*

Proof. Proving this theorem is a good exercise. First we need to figure out what we need to show. We need to show that $\Theta_X^{-1}(\mathcal{E}, \varphi) = \Phi(g, \mathrm{I\!I})$. The inverse of the non-abelian Hodge correspondence is sketched in subsection 3.1: we need to find a Hermitian metric on $\mathcal{E}$ satisfying the self-duality equation (3.2): $F(A) + [\varphi, \varphi^{*_h}] = 0$. Luckily we have a candidate metric h and already know the associated Chern connection, it is A. The curvature of A is computed as

$$F(A) = \begin{pmatrix} F(a^+) - \alpha^* \wedge \alpha & 0 \\ 0 & F(a^-) - \alpha \wedge \alpha^* \end{pmatrix}. \tag{3.10}$$

With $F(a^+) = \frac{i}{2} K_g \,\mathrm{vol}_g = -F(a^-)$ (where K_g is the curvature of g and vol_g its area form) and $\alpha \wedge \alpha^* = \frac{i}{2} \|\alpha\|_g^2 \,\mathrm{vol}_g$, this is written

$$F(A) = \begin{pmatrix} K_g + \|\alpha\|_g^2 & 0 \\ 0 & -K_g - \|\alpha\|_g^2 \end{pmatrix} \frac{i \,\mathrm{vol}_g}{2} \tag{3.11}$$

On the other hand, the bracket $[\varphi, \varphi^{*_h}]$ is computed as

$$[\varphi, \varphi^{*_h}] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{i \,\mathrm{vol}_g}{2} \tag{3.12}$$

Putting things together, the self-duality equation $F(A) + [\varphi, \varphi^{*_h}] = 0$ simply reads $K_g + \|\alpha\|_g^2 + 1 = 0$. Remarkably, this is exactly the Gauss equation (1.6) for the minimal hyperbolic germ $(g, \mathrm{I\!I} = \mathrm{Re}(\alpha))$. This shows that the self-duality equations are satified and $\rho = \Theta^{-1}(\mathcal{E}, \varphi)$ is the holonomy of the flat connection $A + \varphi + \varphi^*$. But in fact we may soon conclude now. Indeed, the ρ-equivariant harmonic map $f_\rho : \tilde{X} \to \mathbb{H}^3$ given by Corlette's theorem must be conformal because $\mathrm{Hopf}(f_\rho) = \mathrm{tr}(\varphi^2) = 0$, so it defines a minimal immersion $\tilde{X} \to \mathbb{H}^3$. According to Donaldson [10], the special form of the Higgs field $(\mathcal{E}, \varphi)$ associated to that harmonic map implies that its first fundamental form is g and its second fundamental form is $\mathrm{I\!I} = \mathrm{Re}(\alpha)$. It follows from the unicity in theorem 1.19.b that ρ is in the conjugacy class of $\Phi(g, \mathrm{I\!I})$. □

One can hope that this explicit description of the map $\Xi : \mathcal{H} \to \mathcal{M}$ opens the way to studying its properties. Here is a first simple example of application taken from [39]. First recall that $\mathcal{H}$ has a natural $U(1)$-action, which by the way preserves the fibers of the projection $\mathcal{H} \to \mathcal{T}(S)$. Note that there is also a bundle $U(1)$-action on $\mathcal{M}$, where $U(1)$ acts in $\mathcal{M}(X)$ by complex multiplication on the Higgs field.

Proposition 3.17. *The map $\Xi : \mathcal{H} \to \mathcal{M}$ is $U(1)$-equivariant.*

Proof. Let $(g, \mathrm{I\!I})$ be a minimal hyperbolic germ on S. By definition, under the action of $e^{i\theta} \in U(1)$, we get a minimal hyperbolic germ $(g_\theta, \mathrm{I\!I}_\theta)$ where $g_\theta = g$ and $\mathrm{I\!I}_\theta$ is the real part of the holomorphic quadratic differential $e^{i\theta}\alpha$. Following the procedure described in the previous theorem, the Higgs bundle $\Xi(g_\theta, \mathrm{I\!I}_\theta)$ is associated to the unitary connection

$$A_\theta = \begin{pmatrix} a^+ & e^{-i\theta}\alpha^* \\ e^{i\theta}\alpha & a^- \end{pmatrix} \tag{3.13}$$

and Higgs field still

$$\varphi_\theta = \varphi = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{3.14}$$

Consider the unitary gauge transformation of E given by

$$U_\theta = \begin{pmatrix} e^{i\frac{\theta}{2}} & 1 \\ 0 & e^{-i\frac{\theta}{2}} \end{pmatrix}. \tag{3.15}$$

Then $U_\theta A_\theta U_\theta^{-1} = A$ and $U_\theta \varphi U_\theta^{-1} = e^{i\theta}\varphi$. This shows that the Higgs bundle $\Xi(g_\theta, \mathrm{I\!I}_\theta)$ associated to A_θ and $\mathrm{I\!I}_\theta$ is gauge-equivalent to the Higgs bundle associated to A and $e^{i\theta}\varphi$, that is $e^{i\theta} \cdot \Xi(g, \mathrm{I\!I})$. Hence Ξ is $U(1)$-equivariant. □

4 Symplectic reduction and moduli spaces

In preparation.

A Hyperkähler structures

This appendix does not have the pretension to be an introduction to hyperkähler manifolds, we merely define some basic concepts and introduce useful notations.

A.1 Kähler structures

As a reminder and to introduce a few concepts and notations for later, we first quickly recall what a Kähler structure is.

Linear Hermitian structures

Definition A.1. Let V be a real vector space. A *linear complex structure* on V is equivalently:

(i) A $\mathbb{C}$-action on V extending the scalar multiplication by reals, giving V the structure of a complex vector space.

(ii) An endomorphism $I \in \mathrm{End}_{\mathbb{R}}(V)$ such that $I^2 = -\mathbf{1}$.

Of course, the relation between the two definitions is that I represents scalar multiplication by i.

Definition A.2. Let V be a real vector space with a linear complex structure. A *linear Hermitian structure* on V is, equivalently:

(i) A Hermitian inner product $h : V \times V \to \mathbb{C}$.
(ii) A real inner product $g : V \times V \to \mathbb{R}$ that is compatible with I in the sense that I is g-orthogonal.

The equivalence between the two definitions is that a Hermitian inner product is written in real and imaginary parts $h = g - i\omega$, where g is a real inner product compatible with I and ω is determined by $\omega(\cdot,\cdot) = g(I\cdot,\cdot)$. Note that ω is a linear symplectic structure on V, *i.e.* a nondegenerate skew-symmetric bilinear pairing, called the *Kähler form* of the Hermitian structure.

Remark A.3. Since the three structures g, I and ω are related by $\omega = g(I\cdot,\cdot)$, two out of three structures determine the third. Consequently, several equivalent definitions of linear Hermitian structures can be given by picking two out of these three structures, provided the appropriate compatibility condition is required. This relates to the *2 out of 3 property* of the unitary group, namely that $U(n)$ is the intersection of any two out of the three groups $O(2n)$, $\mathrm{Sp}(2n,\mathbb{R})$ and $GL(n,\mathbb{C})$.

Kähler structures

Definition A.4.a. Let M be a smooth manifold. An *Kähler structure* on M is the data of a Riemannian metric g and an almost complex structure I such that:
(i) I is g-orthogonal: $g(I\cdot,I\cdot) = g(\cdot,\cdot)$ (*compatibility condition*).
(ii) I is parallel with respect to the Levi-Civita connection of g: $\nabla I = 0$ (*integrability condition*).

When the integrability condition (ii) is not required, (g, I) is called an *almost-Hermitian structure* on M for the following reason. Let $\omega(\cdot,\cdot) = g(I\cdot,\cdot)$ and $h = g - i\omega$. Then h is a Hermitian metric in the tangent bundle (TM, I). The integrability condition may be restated as follows. First note that ω is a nondegenerate 2-form on M (*i.e.* an almost symplectic form). Then $\nabla I = 0$ if and only if I is integrable[46] and ω is closed (so that it is a symplectic structure). Hence the alternate definition:

Definition A.4.b. Let M be a smooth manifold. A Kähler structure on M is the data of a an integrable almost complex structure I on M and a Hermitian metric h on the complex manifold (M, I) such that the *Kähler form* $\omega = -\operatorname{Im}(h)$ is closed.

One can also check that the integrability condition is equivalent to the parallel transport being complex linear in the tangent bundle, so that a Kähler manifold has Riemannian holonomy in $U(n)$ (which is one of the seven groups in Berger's classification, see Theorem A.18).

Following Remark A.3, several equivalent definitions of Kähler structures can be given by picking two out of the three structures g, I and ω, provided the appropriate compatibility and integrability condition is required.

Volume form and Ricci form on a Kähler manifold

In an almost complex manifold (M, I), it is natural to extend tensors complex linearly in the complexification of the tangent bundle $T^{\mathbb{C}}M = TM \otimes_{\mathbb{R}} \mathbb{C}$, which splits as the direct sum of the $+i$- and $-i$

[46] This means that M can be given the (necessarily unique) structure of a complex manifold such that I is the induced almost-complex structure in the tangent space.

eigenspaces of the almost complex structure: $T^{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M$. On Kähler manifolds, there are important relations between the metric and the operators ∂ and $\bar{\partial}$ (the $(1,0)$ and $(0,1)$ parts of the exterior derivative). In particular, there is a specific Hodge theory of the cohomology of compact Kähler manifolds. This is all very classical and we will not expand any further here. However we recall a couple facts about Kähler manifolds that will be used later (especially in subsection A.4.).

Let (M, g, I) be a Hermitian manifold of real dimension $2n$ and denote by ω the Kähler form. Let $K = \Lambda^n\left(\left(T^{1,0}\right)^* X\right)$ denote the canonical line bundle of (M, I). The Kähler form ω is of type $(1,1)$, so its n^{th} exterior power $\omega^{\wedge n}$ is a section of $K \otimes \bar{K}$. It is easy to show that $\frac{1}{n!}\omega^{\wedge n}(\cdot, I\cdot)$ is the metric $\hat{g}$ induced from g in the anticanonical bundle K^*, and that that the volume form of g can be written:

$$\mathrm{vol}_g = \frac{\omega^{\wedge n}}{n!}. \tag{A.1}$$

Recall that the curvature F_h of a holomorphic line bundle $L \to M$ equipped with a Hermitian metric h is the curvature of the Chern connection ∇^h. It is a 2-form on M with values in $\mathrm{End}_{\mathbb{C}} L \approx \mathbb{C}$, which happens to be i times a $(1,1)$-form on M. In a local holomorphic trivialization of L, the Chern connection is written $\nabla^h = d + A$ where $A = \partial \log h$, and the curvature is $F_h = dA = \bar{\partial}\partial \log h$. One can derive from this formula that the cohomology class $\frac{1}{2\pi}[-iF_h] \in H^2(M, \mathbb{R})$ does not depend on h, and it is called the *real first Chern class*[47] of L, denoted $c_1(L)$. The real first Chern class of (M, I) is by definition $c_1(M, I) := -c_1(K) = c_1(K)$.

When I is parallel, *i.e.* (M, g, I) is Kähler, the Ricci curvature tensor[48] Ric of the metric g must preserve the almost complex structure: $\mathrm{Ric}(I\cdot, I\cdot) = \mathrm{Ric}$. One can thus turn Ric into a $(1,1)$-form ω_{Ric} defined by $\omega_{\mathrm{Ric}} = \mathrm{Ric}(I\cdot, \cdot)$, called the *Ricci form* of the Kähler manifold (M, I, g).

Theorem A.5. *The curvature of K^* is equal to $i\,\omega_{\mathrm{Ric}}$.*

This theorem is important in the theory of Kähler manifolds. Together with (A.1), it has in particular the following consequences:

- The Ricci form of a Kähler manifold only depends on the volume form of the metric.
- The Ricci form of a Kähler manifold is closed and its cohomology class not depend on the metric. It is related to the real first Chern class by $[\omega_{\mathrm{Ric}}] = 2\pi c_1(M, I)$.

In light of this it is natural to ask whether, for a complex manifold (M, I) of Kähler type (admitting a Kähler structure), a closed 2-form representing $2\pi c_1(M, I)$ is the Ricci form of some Kähler metric.The answer is provided by the "Calabi conjecture" proved by Yau:

Theorem A.6. *Let (M, I) be a manifold of Kähler type and let ρ be a closed 2-form with $[\rho] = 2\pi c_1(M, I)$. In each Kähler class in $H^2(M, \mathbb{R})$ there is a unique Kähler form whose associated Ricci form is ρ.*

Corollary A.7. *Let (M, I) be a complex manifold of Kähler type with vanishing real first Chern class. Then any Kähler class contains a unique Ricci-flat metric.*

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[47] $c_1(L)$ turns out to be the image of an integral cohomology class in $H^2(M, \mathbb{Z})$, called the first (integral) Chern class of L. Note that this integral class may have torsion, so it contains strictly more information than the real class. It also turns out that $c_1(L)$ does not depend on the holomorphic structure either (one can take any unitary connection to compute it), it is a topological invariant of $L \to M$.

[48] On a Riemannian manifold (M, g), the Ricci curvature tensor is the bilinear form $\mathrm{Ric} : TM \times TM \to \mathbb{R}$ such that $\mathrm{Ric}(x, y)$ is the trace of the endomorphism $w \mapsto R(x, w)y$, where $R(x, y) = [\nabla_x, \nabla_y] - \nabla_{[x,y]}$ is the Riemannian curvature tensor. Symmetries of the Riemannian curvature tensor (Bianchi identities) imply that Ric is symmetric.

A.2 A bit of quaternionic linear algebra

Quaternions

Definition A.8. The algebra of quaternions $\mathbb{H}$ is the unital associative algebra over $\mathbb{R}$ generated by three elements i, j and k satisfying the *quaternionic relations*

$$\begin{aligned} i^2 = j^2 = k^2 = -1 \\ ij = -ji = k \end{aligned} \tag{A.2}$$

$\mathbb{H}$ is a 4-dimensional algebra over $\mathbb{R}$: a generic quaternion is written $q = a + ib + jc + kd$ with $(a, b, c, d) \in \mathbb{R}^4$. Let us give a few classical definitions:

Definition A.9. Let $q = a + ib + jc + kd$ be a quaternion.

- The *real part* of q is the real number $\operatorname{Re} q := a$, its *imaginary part* is the *pure imaginary quaternion* $\operatorname{Im}(q) := ib + jc + kd$ so that $q = \operatorname{Re} q + \operatorname{Im} q$ and accordingly $\mathbb{H} = \mathbb{R} \oplus \operatorname{Im} \mathbb{H}$.
- The *quaternionic conjugate* of q is $\overline{q} = \operatorname{Re} q - \operatorname{Im} q$. Quaternionic conjugation is an involutive antiautomorphism of $\mathbb{H}$: it squares to the identity, is real linear and satisfies $\overline{q_1 q_2} = \overline{q_2}\,\overline{q_1}$.
- The *norm* of q is given by $\|q\|^2 = q\overline{q} = \overline{q}q = a^2 + b^2 + c^2 + d^2$. It is a multiplicative norm on $\mathbb{H}$ and is Euclidean with polarized inner product $\langle q_1, q_2 \rangle = \operatorname{Re}(q_1 \overline{q_2})$, which is the standard inner product on $\mathbb{H} \approx \mathbb{R}^4$. Any nonzero quaternion q has an inverse given by $q^{-1} = \frac{\overline{q}}{\|q\|^2}$, so $\mathbb{H}$ is a division ring.
- Any quaternion q can be written in *polar form* $q = \rho e^{\theta s} = \rho\,(\cos(\theta) + \sin(\theta) s)$, where $\rho = \|q\|$ is a nonnegative real number, s is a unit pure imaginary quaternion and θ is a real number.

Unit quaternions (quaternions with norm 1) form a multiplicative subgroup of $\mathbb{H}^\times$ denoted $\mathrm{Sp}(1)$. Topologically, $\mathrm{Sp}(1)$ is the 3-sphere, since it is the unit sphere in $(\mathbb{H}, \|\cdot\|) \approx \mathbb{R}^4$. We let $S = \mathrm{Sp}(1) \cap \operatorname{Im} \mathbb{H}$ denote the 2-sphere of unit pure imaginary quaternions, which is readily seen to be the set of square roots of -1 in $\mathbb{H}$[49].

Let us consider the action of the multiplicative group $\mathbb{H}^\times = \mathbb{H} - \{0\}$ on $\mathbb{H}$ by conjugation: $g \cdot q = c_g(q) := gqg^{-1}$. This action commutes with quaternionic conjugation, so it preserves the splitting $\mathbb{H} = \mathbb{R} \oplus \operatorname{Im} \mathbb{H}$. Also, it is an isometric action by multiplicativity of the norm. The kernel of the action is the center of $\mathbb{H}$ minus 0, namely $\mathbb{R}^\times$, so one can restrict the action to $\mathrm{Sp}(1)$ without loss and kill most of the kernel, leaving only $\mathbb{R}^\times \cap U = \{\pm 1\}$. The $\mathbb{R}$-part of $\mathbb{H}$ is pointwise fixed. On $\operatorname{Im} \mathbb{H}$, it is straightforward to show to that the action of the unit quaternion $u = e^{\theta s}$ is precisely the Euclidean rotation r_u of oriented axis $\mathbb{R}s$ and angle 2θ. In particular, we get a surjective morphism $c\colon \mathrm{Sp}(1) \to SO(3)$ with kernel $\{\pm 1\}$[50].

Proposition A.10. *Let (s_1, s_2, s_3) be a triple of quaternions. The following are equivalent:*

(i) *(s_1, s_2, s_3) is a direct orthonormal basis of* $\operatorname{Im} \mathbb{H}$.

(ii) *There exists a unit quaternion u (unique up to sign) such that*

$$(s_1, s_2, s_3) = u(i, j, k)u^{-1} = (r_u(i), r_u(j), r_u(k))\,.$$

[49] Note by the way that polynomials may have infinitely many roots in $\mathbb{H}$.

[50] This is maybe the easiest way to see that $SO(3)$ has fundamental group $\mathbb{Z}/2\mathbb{Z}$ and double cover $\mathrm{Sp}(1) = \mathrm{Spin}(3) \approx S^3$.

(iii) (s_1, s_2, s_3) *satisfy the quaternionic relations* (A.2).

We will call a such triple (s_1, s_2, s_3) a *quaternionic triple*.

Remark A.11. Representing quaternions by matrices:

- The multiplication (on the right, say) by a given quaternion defines an element of $\mathrm{End}_{\mathbb{R}} \mathbb{H} \approx \mathcal{M}_{4\times4}(\mathbb{R})$, so any quaternion can be represented by a 4×4 matrix with real entries (which we will not bother writing), and $\mathbb{H}$ can be embedded as a subalgebra of $\mathcal{M}_{4\times4}(\mathbb{R})$.
- If s is any quaternion that squares to -1 (*i.e.* $s \in S$ is a unit pure imaginary quaternion), then multiplication by s on the right (say) is a linear complex structure on $\mathbb{H}$. Moreover, multiplication on the left by a given quaternion commutes with that almost complex structure, so it is a complex linear endomorphism of $(\mathbb{H}, s) \approx \mathbb{C}^2$. Consequently, $\mathbb{H}$ can be embedded as subalgebra of $\mathcal{M}_{2\times2}(\mathbb{C})$. A such representation gives an isomorphism $\mathrm{U}(1, \mathbb{H}) \approx \mathrm{SU}(2, \mathbb{C})$.

Quaternionic vector spaces, linear maps and matrices

Non-commutativity of $\mathbb{H}$ makes quaternionic linear algebra tricky[51]. Many concepts of linear algebra over commutative fields are still valid, for example bases, dimension, many matrix decompositions; however another lot do not work well, including determinants and eigenspaces. It is not our intent to discuss any of that (the curious reader may refer to [37]), we merely want to introduce a few definitions and concepts that are relevant to studying hyperkähler structures.

When discussing quaternionic linear algebra, it has to be decided whether we deal with left or right vector spaces. Here we are faced with a dilemma. On the one hand, right vector spaces work better with matrices, because the relation between vectors and linear maps to matrices is the same as in classical linear algebra[52]. On the other hand, in complex differential geometry, the common usage is to represent (almost) complex structures by linear endomorphisms, which represent scalar multiplication on the left. Of course, the theory of left and right vector spaces are equivalent, but in practice this can be a headache. Since we are on the differential geometry side, we will stick to left quaternionic vector spaces and deal with the slightly unpleasant consequences when working with matrices [53].

Definition A.12. A *left quaternionic vector space* is equivalently:

(i) A left $\mathbb{H}$-module V.

(ii) A real vector space V with a *linear quaternionic structure*, that is the data of three endomorphisms $I, J, K \in \mathrm{End}_{\mathbb{R}}(V)$ that satisfy the quaternionic relations (A.2) under composition.

Of course, the relation between the two definitions is that I is the scalar multiplication by i, etc.

Remark A.13. A couple remarks about linear quaternionic structures:

[51] Quaternionic analysis is arguably worse, see *e.g.* [46].

[52] Namely: with respect to a basis, a vector may be represented as a column vector with quaternion entries. The action of linear map is represented by matrix multiplication on the left, and composition of linear maps corresponds to matrix multiplication in the same order as we write the composition. In particular, choosing a basis of an n-dimensional right quaternionic vector space V gives an isomorphism of algebras $\mathrm{End}_{\mathbb{H}} V \xrightarrow{\sim} \mathcal{M}_{n\times n}(\mathbb{H})$.

[53] A maybe more ambitious solution would be to embrace a different convention for hyperkähler structures as Bryant does in [14], although that goes against quasi-unanimous usage.

- I, J and K are all linear complex structures on V, since they square to $-\mathbf{1}$. More generally, if $s = bi + cj + dk \in S$ is a unit pure imaginary quaternion (*i.e.* a square root of -1), then $I_s := bI + cJ + dK$ is a linear complex structure on V. Hence V is equipped with a 2-sphere of linear complex structures parametrized by S.
- If (s_1, s_2, s_3) is a quaternionic triple, then $(I_{s_1}, I_{s_2}, I_{s_3})$ is a new linear quaternionic structure on V. However it defines the same 2-sphere of linear complex structures, equivalently it spans the same subspace of $\operatorname{End}_{\mathbb{R}} V$ (isomorphic to $\operatorname{Im}\mathbb{H}$). Two such linear quaternionic structures on V will be called *equivalent*. From the discussion of the previous paragraph, equivalent linear quaternionic structures are parametrized by $SO(3)$.

Now let us discuss linear maps between quaternionic vector spaces. Just for comfort, we restrict our attention to the case where the domain and target space are the same.

Definition A.14. Let $f : V \to V$ be a map from a left quaternionic vector space V to itself. f is called *quaternionic linear* if, equivalently:
(i) f is a morphism of left $\mathbb{H}$-modules.
(ii) f is real linear and commutes with I, J and K.

Let $(e_j)_{1\leqslant j\leqslant n}$ be a basis of V (we assume that V is finite-dimensional). Then any vector $x \in V$ can be written $x = x_1 e_1 + \cdots + x_n e_n$, where $x_j \in \mathbb{H}$. By linearity, $f(x) = \sum_k x_k f(e_k)$, so $f(x) = \sum_{j,k} x_k a_{kj} e_j$ where a_{ij} are the quaternions such that $f(e_i) = \sum_j a_{ij} e_j$. Let X denote the *row* vector $X = (x_1, \ldots, x_n)$ and A the $n \times n$ matrix with (i, j)-entry a_{ij}. Be wary that A is the *transpose* of the matrix that we would define in classical linear algebra. The computation above shows:

Proposition A.15. *Let $X, Y \in \mathbb{H}^{1\times n}$ be the row vectors associated to $x, y \in V$ and $A = A_f \in \mathcal{M}_{n\times n}(\mathbb{H})$ be the matrix associated to a linear map $f : V \to V$ as above. Then*

$$y = f(x) \;\Leftrightarrow\; Y = X\,A_f\,. \tag{A.3}$$

It follows that the map $\operatorname{End}_{\mathbb{H}} V \to \mathcal{M}_{n\times n}(\mathbb{H})$ defined by $f \mapsto A_f$ is an anti-isomorphism of algebras:

$$A_{f\circ g} = A_g\,A_f\,. \tag{A.4}$$

An alternative approach is the following. Represent vectors x, y by column vectors X, Y as usual, also define the matrix A_f associated to a linear map f the usual way. Define a new matrix multiplication[54] in $\mathcal{M}_{n\times n}(\mathbb{H})$ by

$$A \bullet B = \Big(\sum_k b_{ik} a_{kj}\Big)_{1\leqslant i,j\leqslant n} = {}^t\big({}^tB\,{}^tA\big) \tag{A.5}$$

where $M \mapsto {}^tM$ denotes matrix transposition. Then $Y = A \bullet X$ and $\operatorname{End}_{\mathbb{H}} V \to (\mathcal{M}_{n\times n}(\mathbb{H}), \bullet)$ is an isomorphism of algebras.

Of course, once a basis has been chosen, an n-dimensional left quaternionic vector space V is isomorphic to $\mathbb{H}^{1\times n}$, with scalar multiplication given by entry-wise quaternion multiplication on the left. An additive map $f : \mathbb{H}^{1\times n} \to \mathbb{H}^{1\times n}$ is linear if and only if it commutes with scalar multiplication, and we saw that such maps are given by matrix multiplication on the right.

Naturally, the group of invertible quaternionic linear endomorphisms of V is denoted $\mathrm{GL}_{\mathbb{H}}(V)$, and the group of invertible matrices in $\mathcal{M}(\mathbb{H})$ is denoted $\mathrm{GL}(n, \mathbb{H}) \approx \mathrm{GL}_{\mathbb{H}}(\mathbb{H}^n)$.

[54] This is the "right multiplication" discussed in [41].

Linear hyper-Hermitian structures

Let V be a left quaternionic vector space. Denote by (I, J, K) the linear quaternionic structure on V.

Definition A.16. A *linear hyper-Hermitian (or hyperkähler) structure* on V is, equivalently, the data of:

(i) A *hyper-Hermitian inner product*, that is a pairing $H : V \times V \to \mathbb{H}$ such that (1) H is quaternionic linear in the first slot (2) H has hyper-Hermitian symmetry: switching the arguments produces the quaternionic conjugate and (3) $H(x, x)$ is positive definite.

(ii) A real inner product $g : V \times V \to \mathbb{R}$ such that I, J and K are g-orthogonal.

The relation between the two definitions is that a hyper-Hermitian pairing H is written in real and imaginary parts $H = g - \omega_h$, where ω_h is determined by g and I, J, K:

$$\omega_h = i\omega_I + j\omega_J + k\omega_K = ig(I\cdot, \cdot) + jg(J\cdot, \cdot) + kg(K\cdot, \cdot)\,. \tag{A.6}$$

Note that (g, I), (g, J) and (g, K) are linear Hermitian structures with associated Kähler forms $\omega_I, \omega_J, \omega_K$. More generally, for any unit pure imaginary quaternion $s \in S$, we get a linear Hermitian structure (g, I_s) with Kähler form $\omega_s = g(I_s\cdot, \cdot)$.

ω_h is called (by us) the *hyperkähler form* of the hyper-Hermitian hermitian structure. It is a skew-symmetric real bilinear operator with values in $\operatorname{Im}\mathbb{H}$:

$$\omega_h \in \Lambda^2 V^* \otimes_{\mathbb{R}} \operatorname{Im}\mathbb{H}\,. \tag{A.7}$$

Be cautious here that V^* denotes the *real* dual of V[55].

A small computation shows that $\omega_c := \omega_J + i\omega_K$ is a linear complex symplectic structure on the linear complex vector space (V, I) (it is a nondegenerate, I-complex bilinear, skew-symmetric form). In this sense, a linear hyper-Hermitian structure may be considered as a refinement of a linear complex symplectic structure. More generally, if (s_1, s_2, s_3) is any quaternionic triple, then $\omega^c_{(s_1,s_2,s_3)} := \omega_{s_2} + i\omega_{s_3}$ is a linear complex symplectic structure on (V, I_{s_1}). If s_2 and s_3 are not specified, then $\omega^c_{(s_1,s_2,s_3)}$ is defined up to a multiplicative unit complex number.

A quaternionic linear endomorphism of V is called *(hyper)unitary* if it preserves H, or equivalently g. The group of quaternionic unitary endomorphisms of V is denoted $\mathrm{U}(V, H)$ or $\mathrm{Sp}(V, H)$. It can be described as the intersection $\mathrm{Sp}(n) = \mathrm{End}_{\mathbb{H}}(V) \cap \mathrm{O}(V, g)$.

The canonical example of a hyper-Hermitian linear structure is the left quaternionic vector space $\mathbb{H}^n$ with the hyper-Hermitian inner product $H(x, y) = x_1\overline{y_1} + \cdots + x_n\overline{y_n} = XY^*$, where $M \mapsto M^*$ denotes the transpose quaternionic conjugation of a matrix (here a row vector). Transpose conjugation reverses products: $(AB)^* = B^*A^*$. The matrix subgroup of $GL_n(\mathbb{H})$ corresponding to quaternionic unitary endomorphisms of $(\mathbb{H}^n, H)$ is denoted $\mathrm{Sp}(n)$[56] or $\mathrm{U}(n, \mathbb{H})$, it is characterized as

$$\mathrm{Sp}(n) = \{A \in \mathcal{M}_{n\times n}(\mathbb{H}),\ A\,A^* = \mathbf{1}\} = \mathrm{GL}_n(\mathbb{H}) \cap \mathrm{O}(4n)\,. \tag{A.8}$$

[55] We will not work with the quaternionic dual of V, which by the way is a *right* quaternionic vector space, making transposition a bit tricky.

[56] This standard notation is potentially misleading: $\mathrm{Sp}(n)$ is not a symplectic group. It is, however, the maximal compact in the complex symplectic group $\mathrm{Sp}(2n, \mathbb{C})$. Under the identification $\mathbb{H}^n \approx \mathbb{C}^{2n}$, $\mathrm{Sp}(n) = \mathrm{Sp}(2n, \mathbb{C}) \cap \mathrm{U}(2n, \mathbb{C})$. For this reason, the quaternionic unitary group $\mathrm{Sp}(n)$ is commonly called the *compact symplectic group*.

A.3 Hyperkähler structures

Definitions

Similarly to a Kähler structure, a hyperkähler structure on a manifold is a smooth family of linear hyper-Hermitian structures in its tangent spaces, with an additional integrability condition:

Definition A.17.a. Let M be a smooth manifold. An *hyperkähler structure* on M is the data of a Riemannian metric g and three almost complex structure I, J and K such that:

(i) I, J and K satisfy the quaternionic relations: $IJ = -JI = K$.

(ii) I, J and K are g-orthogonal (*compatibility condition*).

(iii) I, J and K are parallel with respect to the Levi-Civita connection of g: $\nabla I = \nabla J = \nabla K = 0$ (*integrability condition*).

Note that conditions (ii) and (iii) are precisely saying that (g, I), (g, J) and (g, K) are Kähler structures on M. Naturally, we let ω_I, ω_J and ω_K denote the respective Kähler forms.

We give an equivalent definition in accordance with our discussion in the linear setting:

Definition A.17.b. Let M be a smooth manifold. A *hyperkähler structure* on M is the data of:

(i) A quaternionic structure in the tangent bundle of M, *i.e.* a smooth bundle action of $\mathbb{H}$ in TM, making each tangent space T_xM a left quaternionic vector space,

(ii) A hyper-Hermitian metric on M, *i.e.* a smooth pairing $H : TM \times TM \to \mathbb{H}$ such that H is a hyper-Hermitian inner product in each tangent space T_xM,

such that the the *hyperkähler form* $\omega_h := -\operatorname{Im} H \in \Lambda^2(M, \operatorname{Im}\mathbb{H})$ is closed (*integrability condition*).

A manifold with just a quaternionic structure in tangent bundle (condition (i)) is usually called *almost hypercomplex*. Of course, this is the same as three almost complex structures I, J and K satisfying the quaternionic relations. A hyper-Hermitian metric H on M is then equivalent to a Riemannian metric g on M that is compatible with I, J and K, as we saw in the previous subsection. The hyperkähler form ω_h is given in terms of the Kähler forms of (g, I), (g, J) and (g, K) by $\omega_h = i\omega_I + j\omega_J + k\omega_K$. Nevertheless, there is something nontrivial about the equivalence of the two definitions: it is surprising that the integrability condition $d\omega_h = 0$ is enough. It is just saying that ω_I, ω_J and ω_K are closed, and one would expect the additional requirement that I, J and K are integrable. Hitchin showed in [18] that this is automatically satisfied. This fact is useful in the setting of hyperkähler reduction.

The 2-sphere of Kähler structures

Let (M, g, I, J, K) be a hyperkähler manifold. Recall that S denotes the Euclidean 2-sphere of unit pure imaginary quaternions (*i.e.* square roots of -1 in $\mathbb{H}$).

For any $s = bi + cj + dk \in S$, $I_s = bI + cJ + dK$ is an almost complex structure on M, which is clearly parallel with respect to g and easily checked to be g-orthogonal, so (g, I_s) is a Kähler structure on M. A hyperkähler manifold is thus equipped with a 2-sphere of Kähler structures $\{(g, I_s),\ s \in S\}$.

Consider a fixed identification $S \approx \mathbb{C}\mathbf{P}^1$ given by stereographic projection. Let Z be the complex manifold $Z := M \times \mathbb{C}\mathbf{P}^1$, where the complex structure on Z is induced by the complex structures

I_s on the horizontal slices $M \times \{s\}$ and by the complex structure of $\mathbb{C}\mathbf{P}^1$ on the vertical slices. The projection on the second factor $p : Z \to \mathbb{C}\mathbf{P}^1$ is clearly a holomorphic fiber bundle, called the *twistor space* of the hyperkähler manifold M. The twistor space encapsulates all of the hyperkähler structure on M in terms of holomorphic data: given a holomorphic fiber bundle $p : Z \to \mathbb{C}\mathbf{P}^1$ satisfying the appropriate conditions, it is possible to recover the hyperkähler structure in the typical fiber. We refer to [20] for details.

Note that when (s_1, s_2, s_3) is any quaternionic triple and $c > 0$ is a positive real number, then $(cg, I_{s_1}, I_{s_2}, I_{s_3})$ is a new hyperkähler structure on M. Two such hyperkähler structures are called *equivalent*. Two hyperkähler structures are equivalent if and only if their Riemannian metrics are constant proportional and their triple of complex structures generate the same subspace of $\mathrm{End}_{\mathbb{R}}(TM)$. Presumably, this amounts to saying that their twistor spaces are isomorphic (I have not carefully checked).

As expected from the linear algebra, given a hyperkähler manifold (M, g, I, J, K), the complex-valued 2-form $\omega_c := \omega_J + i\omega_K$ is a complex symplectic form on the complex manifold (M, I). In this sense a hyperkähler structure is a refinement of a complex symplectic structure. More generally, if (s_1, s_2, s_3) is any quaternionic triple, then $\omega^c_{(s_1,s_2,s_3)} := \omega_{s_2} + i\omega_{s_3}$ is a complex symplectic structure on (M, I_{s_1}). If s_2 and s_3 are not specified, then $\omega^c_{(s_1,s_2,s_3)}$ is defined up to a multiplicative unit complex number.

Riemannian holonomy

Let (M, g) be a connected Riemannian manifold of dimension n. Given a basepoint $p \in M$, parallel transport along loops based at p defines a subgroup of $\mathrm{O}(T_pM) \approx \mathrm{O}(n)$ called the *Riemannian (full) holonomy group* of (M, g) at p. Ignoring the choice of the basepoint p and the linear isometric isomorphism $T_pM \approx \mathbb{R}^n$, the holonomy group is a subgroup of $\mathrm{O}(n)$ defined up to conjugation. The *restricted holonomy group* is obtained by parallel transporting only along null-homotopic loops[57]. A generic Riemannian manifold has (restricted) holonomy group $\mathrm{O}(n)$ if it is nonorientable and $SO(n)$ if it is orientable. When the (restricted) holonomy group turns out to be a proper subgroup, the Riemannian manifold (M, g) is said to have *special holonomy*.

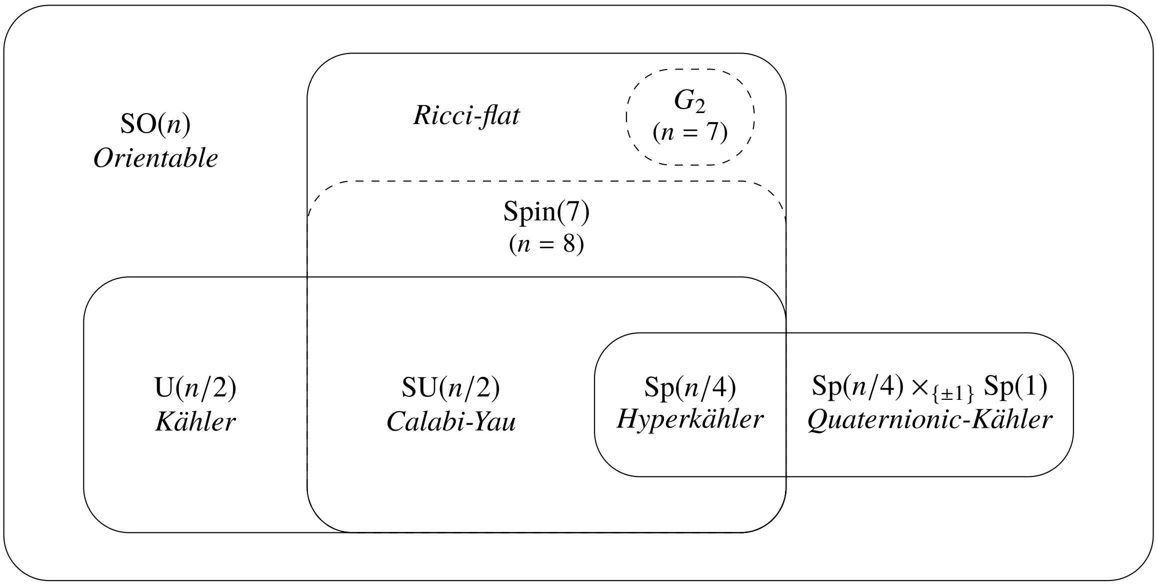
Let now (M, g, I, J, K) be a hyperkähler manifold. Since I, J and K are parallel with respect to the Levi-Civita connection of g, parallel transport commutes with quaternionic multiplication. In particular, the Riemannian holonomy group of (M, g) is contained in $\mathrm{GL}_n(\mathbb{H}) \cap \mathrm{O}(4n) = \mathrm{Sp}(n)$. Conversely, given a Riemannian manifold (M, g) with holonomy in $\mathrm{Sp}(n)$, one can construct a hyperkähler structure by parallel transporting three adequate linear complex structures I_p, J_p, K_p in some tangent space T_pM. However, this construction may give rise to several inequivalent hyperkähler structures. Some authors still define a hyperkähler manifold as a Riemannian manifold with holonomy in $\mathrm{Sp}(n)$, which can be a source of (mild) confusion.

Let us recall Berger's classification of Riemannian holonomy groups:

Theorem A.18. *Let (M, g) be an orientable complete Riemannian manifold of dimension n which is irreducible (not locally a product) and nonsymmetric (not locally a Riemannian symmetric space).*

[57] Equivalently (though this is not trivial), the restricted holonomy group is the identity component of the full holonomy group.

Then the restricted holonomy group of (M, g) is one of the seven groups in the following diagram:



Thus hyperkähler manifolds are "the most special" Riemannian manifolds: their holonomy group lies in the intersection of all special holonomy groups (setting aside the exceptional case G_2), as shown in the diagram above[58]. Note that Riemannian manifolds with special holonomy can basically be classified in three families: Kähler, Ricci-flat and quaternionic. Hyperkähler manifolds are the ones that belong to all three families.

A.4 Hyperkähler structures and Calabi-Yau structures

There are several inequivalent definitions of Calabi-Yau manifolds in the literature. We give one that is most adapted to our discussion. For the interested reader, a nice overview of Calabi-Yau manifolds is layed down by Yau in the scholarpedia article [53].

Definition A.19. Let M be a smooth connected manifold[59] of dimension $2n$. A *Calabi-Yau package* on M is the data of:

(i) A Kähler structure (g, I).

(ii) A holomorphic volume form Ω on (M, I), *i.e.* a closed $(n, 0)$ form, *i.e.* a holomorphic section of the canonical bundle of (M, I), which satisfies the "normalizing" condition:

$$c_n \Omega \wedge \bar{\Omega} = \mathrm{vol}_g \tag{A.9}$$

[58]This relates to the fact that $\mathbb{H}$ is the largest normed division algebra over the reals (Frobenius theorem), setting aside the nonassociative algebra of octonions (which actually gives rise to the two exceptional cases Spin(7) and G_2).

[59]Usually Calabi-Yau manifolds are required to be compact, but we do not make that assumtion.

where c_n is a constant given by $c_n = \left(\frac{i}{2}\right)^n (-1)^{\frac{n(n-1)}{2}}$ [60].

Proposition A.20. *Let (M, g, I, Ω) be a Calabi-Yau manifold. Then:*

(i) *g is Ricci-flat.*

(ii) *(M, I) has vanishing real first Chern class: $c_1(M, I) = 0$.*

(iii) *g has holonomy in $SU(n)$.*

(iv) *Ω is parallel: $\nabla^g \Omega = 0$.*

These properties and their relations can be derived our previous discussion about the Ricci curvature on a Kähler manifold (last part of subsection A.1). For (iv), the additional ingredient required is Bochner's formula[61]. These properties are interrelated and each one of them defines a Calabi-Yau structure on M in some sense, although one may have to change the metric or the holomorphic volume form. We leave the details of this discussion to the reader as a (nice) exercise.

A.5 Examples

Examples of hyperkähler structures are not easy to come by. Quaternionic structures are already hard to construct: there is no quaternionic equivalent of complex differential geometry, because the notion of quaternionic differentiability is too restrictive. Kähler structures are "abundantly" provided by complex submanifolds of the complex projective space, by contrast the quaternionic projective space $\mathbb{H}\mathbf{P}^n$ itself is not hyper-Kähler, it is only "quaternionic-Kähler" (with holonomy in $\mathrm{Sp}(n) \times_{\{\pm 1\}} \mathrm{Sp}(1)$).

Eguchi-Hanson metric

In preparation.

Compact hyperkähler manifolds of real dimension 4

In preparation.

Hyperkähler reduction

In preparation.

Feix hyperkähler structure on the cotangent bundle of Kähler manifold

In preparation.

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[60] This constant c_n is chosen to make $\mathrm{Re}\,\Omega$ a *calibration*. It also gives Ω have a nice expression in suitable coordinates: the prototype of a Calabi-Yau manifold is $\mathbb{C}^n$ with its standard flat Kähler structure and holomorphic volume form $\Omega = dz_1 \wedge \cdots \wedge dz_n$.

[61] Bochner's formula implies that holomorphic tensors on a Ricci-flat compact Kähler manifold are always parallel. However in this situation, compactness of M is not needed.

B Symplectic reduction

A reference : l'autre conne dans téléchargements. Nice reference for symp reduction : Marsden-Weinstein [33] Historically, symplectic reduction has been used to reduce the number of dimensions to study Hamiltonian systems, but this is not our approach here. We will use symplectic reduction to study the structure of moduli spaces, which has also been a spectacular use of symplectic reduction in mathematics and mathematical physics.

B.1 Hamiltonian actions on symplectic manifolds

Let (M, ω) be a connected symplectic manifold. Recall that the *symplectic form* ω is a closed non-degenerate 2-form on M. The non-degeneracy of ω is expressed as the fact that the "symplectic duality" map

$$\begin{aligned} \omega^\flat : TM &\to T^*M \\ v &\mapsto i_v\omega = \omega(v, \cdot) \end{aligned}$$

is invertible. We let $\omega^\sharp$ denote its inverse. A vector field V is called *symplectic* if the dual one-form $i_V\omega$ is closed, and it is called *Hamiltonian* when $i_V\omega$ is exact. It is an immediate consequence of "Cartan's magic formula" $\mathcal{L}_V = i_V \circ d + d \circ i_V$ that V is a symplectic vector field if and only if it preserves the symplectic structure in the sense that $\mathcal{L}_V\omega = 0$ (here $\mathcal{L}_V$ is the Lie derivative). A standard computation shows that this amounts to saying that the flow of V acts by symplectomorphisms. When V is a Hamiltonian vector field, there exists a function $f : M \to \mathbb{R}$ (unique up to addition of a constant) such that $df = i_V\omega$, in that case $V = \omega^\sharp(df)$ is called the *Hamiltonian vector field* (or *symplectic gradient*) of f, often denoted X_f, and f is called a *Hamiltonian function for* V. The *Poisson bracket* of two functions f and g is defined by $\{f, g\} = \omega(X_f, X_g)$, giving $C^\infty(M)$ the structure of a Lie algebra[62].

Now assume that a connected Lie group G acts smoothly on M. The infinitesimal action of G is the map

$$\begin{aligned} V : \mathfrak{g} &\to \Gamma(TM) \\ \xi &\mapsto V_\xi \end{aligned}$$

defined by $(V_\xi)_x = \frac{d}{dt}_{|t=0}(e^{t\xi} \cdot x)$ for all $x \in M$. Here we have denoted $\mathfrak{g}$ the Lie algebra of G and $\Gamma(TM)$ the Lie algebra of vector fields on M. Note that V is always a homomorphism of Lie algebras. The action of G on (M, ω) is called *symplectic* if G acts by symplectomorphisms. By the discussion above this equates to V_ξ being a symplectic vector field for all $\xi \in \mathfrak{g}$. When V_ξ is in fact always a Hamiltonian vector field, the action of G is called *almost Hamiltonian*[63] (this is sometimes automatically the case -for instance when M has vanishing first cohomology). In other words, the

[62] Moreover the Poisson bracket acts as a derivation on the product of functions, giving $C^\infty(M)$ the structure of a *Poisson algebra* (see [23] for a precise definition).

[63] Some authors call it *weakly Hamiltonian*, some just *Hamiltonian* -creating potential confusion.

action of G is almost Hamiltonian when there is a map $H : \mathfrak{g} \to C^\infty(M), \xi \mapsto H_\xi$ such that $V_\xi = X_{H_\xi}$ for all $\xi \in \mathfrak{g}$. It is easy to check that H can always be chosen linear, and this choice is unique up to addition of a linear map $\sigma : \mathfrak{g} \to \mathbb{R}$. The action of G is called *(strongly) Hamiltonian* (or *Poisson*) when σ can be chosen to make H a homomorphism of Lie algebras for the Poisson bracket on $C^\infty(M)$. Sometimes any almost Hamiltonian G-action on M is automatically Hamiltonian, for instance when $\mathrm{H}^2(\mathfrak{g}, \mathbb{R}) = 0$; in any case when the action is Hamiltonian the choice of σ is unique up to an element of $\mathrm{H}^1(\mathfrak{g}, \mathbb{R}) = \{\sigma \in \mathfrak{g}^*, [\mathfrak{g}, \mathfrak{g}] \subset \ker \sigma\}$[64]. Note that when G is semisimple $\mathrm{H}^1(\mathfrak{g}, \mathbb{R}) = \mathrm{H}^2(\mathfrak{g}, \mathbb{R}) = 0$ (*Whitehead lemma*), whence any almost Hamiltonian G-action is Hamiltonian and the choice of H is unique.

B.2 Moment maps

Given an almost Hamiltonian action of a connected Lie group G on a connected manifold M and a choice of a linear $H : \mathfrak{g} \to C^\infty(M)$ such that $V_\xi = X_{H_\xi}$ for all $\xi \in \mathfrak{g}$, the *moment map*[65] for the action is the map

$$\begin{aligned} \mu \colon M &\to \mathfrak{g}^* \\ x &\mapsto [\xi \mapsto H_\xi(x)] \end{aligned}$$

in other words $\langle \mu(\cdot), \xi \rangle = H_\xi$. A moment map is characterized by the relation

$$d\langle \mu(\cdot), \xi \rangle = i_{V_\xi}\omega \tag{B.1}$$

for all $\xi \in \mathfrak{g}$. This could be rewritten $d\mu = \omega^\flat(V)$, in this sense *μ is a Hamiltonian function for the infinitesimal action*. A small computation shows that μ is G-equivariant (where G acts on $\mathfrak{g}^*$ by the coadjoint action) if and only if H is a homomorphism of Lie algebras. By the preceding discussion,

Proposition B.1. *Let G be a connected Lie group acting on a connected symplectic manifold (M, ω) by symplectomorphisms.*

(i) There exists a moment map $\mu \colon M \to \mathfrak{g}^*$ if and only if the action is almost Hamiltonian. A moment map is unique up to addition of a constant $\sigma \in \mathfrak{g}^*$.

(ii) There exists an equivariant moment map if and only if the action is Hamiltonian[66]. *An equivariant moment map is unique up to addition of a constant $\sigma \in \mathrm{H}^1(\mathfrak{g}, \mathbb{R}) \subset \mathfrak{g}^*$.*

Example B.2. For the reader's convenience, we give a few typical Hamiltonian actions and moment maps:

[64] Let us develop a bit further for the thorough reader. One can associate to an almost Hamiltonian action the map $\tau : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ given by $\tau(\xi, \eta) = \{H_\xi, H_\eta\} - H_{[\xi,\eta]}$. This map is a 2-cocycle in the cochain complex $C^\bullet(\mathfrak{g}, \mathbb{R})$ in the sense of Lie algebra cohomology. The class $[\tau] \in \mathrm{H}^2(\mathfrak{g}, \mathbb{R})$ does not depend on the choice of the additive constant σ in H. The action is Poisson exactly when $[\tau] = 0$, which means that $\tau = d\sigma$ for some 1-cochain σ (concretely σ is a linear map $\mathfrak{g} \to \mathbb{R}$ and $\tau(\xi_1, \xi_2) = \sigma([\xi_1, \xi_2])$); choosing a such σ for the additive constant in H makes it a Lie algebra homomorphism. In particular every almost Hamiltonian action is Hamiltonian when $\mathrm{H}^2(\mathfrak{g}, \mathbb{R}) = 0$. A finer obstruction is given by the *equivariant cohomology* $H^2_G(M)$, but this is beyond our scope.

[65] The terminology *momentum map* is also used and arguably more correct as the English translation of the French *application moment* coined by Souriau [45].

[66] G-equivariance is often enforced in the definition of a moment map (in fact Donaldson does in [10]).

(1) **$\mathbb{R}$-actions.** Let $G = \mathbb{R}$. A symplectic action of G on (M, ω) is the flow of a complete symplectic vector field V. The action is (almost) Hamiltonian if and only if V is a Hamiltonian vector field, and an equivariant moment map is a function $f : M \to \mathrm{Lie}(\mathbb{R})^* \approx \mathbb{R}$ that is a Hamiltonian function for V.

(2) **Cotangent bundle action.** Let a connected Lie group G act smoothly on a connected manifold X, denote by $V : \mathfrak{g} \to \Gamma(TX)$ the infinitesimal action. There is natural induced bundle action of G on the cotangent bundle $M = T^*X$, and this action is Hamiltonian with respect to the canonical symplectic structure ω_c. An equivariant moment map $\mu : T^*X \to \mathfrak{g}^*$ is given by $\langle \mu(\alpha), \xi \rangle = \alpha(V_\xi)$.

(3) **Coadjoint orbits.** Let G be a compact connected Lie group and let $M = G \cdot \alpha \subset \mathfrak{g}^*$ be a coadjoint orbit[67]. M is equipped with a symplectic structure as follows. The tangent space to M at a point $\beta = g \cdot \alpha \in \mathfrak{g}^*$ is given by $T_\beta M = \{\mathrm{ad}^*_\xi \beta,\ \xi \in \mathfrak{g}\}$. The symplectic form at β is defined by $\omega(\mathrm{ad}^*_\xi \beta, \mathrm{ad}^*_{\xi'} \beta) = \langle \beta, [\xi, \xi'] \rangle$[68]. The coadjoint action of G (or any subgroup) on M is Hamiltonian, and an equivariant moment map is given by the inclusion $M \to \mathfrak{g}^*$.

(4) **Angular momentum.** Let $G = SO(3)$ act on $\mathbb{R}^3$ by linear isometries. As a special case of example (2), G naturally acts symplectically on $M = T^*\mathbb{R}^3$. Under the identification $(\mathbb{R}^3)^* \approx \mathbb{R}^3$ given by the standard inner product on $\mathbb{R}^3$, the action of G on $M \approx \mathbb{R}^3 \times \mathbb{R}^3$ is given by $A \cdot (q, p) = (Aq, Ap)$. Recall that the Lie algebra $\mathfrak{so}(3)$ is isomorphic to $(\mathbb{R}^3, \times)$ (where $\times$ denotes the cross-product on $\mathbb{R}^3$)[69]. Using these identifications, the equivariant moment map $\mu : M \to \mathfrak{so}(3)^*$ is the map $\tilde{\mu} : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ given by $\mu(q, p) = q \times p$. In classical physics, M can be described as the phase space of a particle, and μ is called its angular momentum.

B.3 Symplectic reduction

Let (M, ω) be a connected symplectic manifold with a Hamiltonian action of a connected Lie group G, and let $\mu : M \to \mathfrak{g}^*$ be an equivariant moment map for the action. For $\alpha \in \mathfrak{g}^*$, denote by $M_\alpha = \mu^{-1}(\alpha) \subset M$ the α-level set of μ and denote by G^0_α the identity component of the stabilizer $G_\alpha \subset G$ for the coadjoint action of G on $\mathfrak{g}^*$. By equivariance of μ, the action of G^0_α on M preserves M_α. Now assume that

(i) α is a regular value[70] of μ, so that M_α is a submanifold of M by the submersion theorem.

(ii) The action of G^0_α on M_α is free and proper, so that the quotient M_α / G^0_α has a natural smooth structure making the projection $p : M_\alpha \to M_\alpha / G^0_\alpha$ a smooth map.

The following theorem defining symplectic reduction is due to Marden-Weinstein [32]:

Theorem B.3. *Under the assumptions above*[71]*, the quotient manifold M_α / G^0_α enjoys a unique sym-*

[67] where G acts on $\mathfrak{g}^*$ via the coadjoint action $g \cdot \alpha = \mathrm{Ad}^*_g \alpha = \alpha \circ \mathrm{Ad}_g$.

[68] This is called the Kirillov-Kostant-Souriau symplectic structure. Its closedness derives from the Jacobi identity.

[69] The isomorphism $\mathfrak{so}(3) \to \mathbb{R}^3$ is given by $\begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \mapsto (a_1, a_2, a_3)$.

[70] Recall that a regular value of a smooth map $f : M \to N$ is a point $y \in N$ such that $df_{|x} : T_x M \to T_y N$ is surjective for all preimages x of y. We assume that $f^{-1}(y) \neq \emptyset$.

[71] The assumptions (i) and (ii) may be slightly weakened as follows. (i'): α is a *clean value*. This means that $M_\alpha = \mu^{-1}(\alpha)$

plectic structure ω_α such that the projection map $p : M_\alpha \to M_\alpha/G_\alpha^0$ is a "symplectomorphism"[72].

The symplectic manifold $(M_\alpha/G_\alpha^0, \omega_\alpha)$ is called a *symplectic reduction* (or *symplectic quotient*, or *Marsden-Weinstein quotient*) of (M, ω).

When α happens to be a fixed point of the coadjoint action, note that the full group G preserves M_α and the quotient is taken with respect to the full group action (since $G = G_\alpha^0$). Moreover, α is a fixed point of the coadjoint action if and only if $\alpha \in \mathrm{H}^1(\mathfrak{g}, \mathbb{R})$ if and only if the shifted moment map $\tilde{\mu} = \mu - \alpha$ is again equivariant. This *shift trick* allows one to define symplectic reduction at $\alpha = 0$ without loss of generality[73]. When it is understood which value of α allowing symplectic reduction is chosen, the symplectic quotient M_α/G is often called "the" symplectic reduction of M and denoted $M/\!/G$.

Remark B.4. It is not hard to show that regular points of μ are the points of M with discrete stabilizer. For this reason the quotient $\mu^{-1}(\alpha)/G$ is never very bad when α is a regular value (G_α^0 acts locally freely on $\mu^{-1}(\alpha)$ so the quotient has an orbifold structure). When α is not a regular value, one can try to work with the quotient $\mu^{-1}(\alpha)_{\mathrm{reg}}/G_\alpha^0$ (where $\mu^{-1}(\alpha)_{\mathrm{reg}}$ denotes the set of regular points in $\mu^{-1}(\alpha)$). It is of course always possible to take the full quotient $\mu^{-1}(\alpha)/G_\alpha^0$, but it may have more serious singularities.

Example B.5. Let us give a first couple of examples of symplectic reduction.

(1) Let us look back at example (2) in the previous set of examples B.2 for moment maps: consider the Hamiltonian action of G on a cotangent bundle T^*X induced by an action of G on X. Assume that the action of G on X is free and proper. Check that the 0-level set of the moment map $\mu : T^*X \to \mathfrak{g}^*$ is the subbundle $T_v^\perp X \subset T^*X$ of covectors that vanish on the vertical tangent bundle $T_v X \subset TX$ to the bundle projection $X \to X/G$. The action of G on $T_v^\perp X$ is free and proper and the projection $T_v^\perp X \to T^*(X/G)$ induces a symplectic isomorphism

$$T^*X/\!/G \xrightarrow{\sim} T^*(X/G)\,. \tag{B.2}$$

Details are left to the reader.

(2) Let $(M, \omega) = \mathbb{C}^n$ where ω is the standard symplectic structure on $\mathbb{C}^n$ given by $\omega = -\operatorname{Im}(h)$, where h is the usual Hermitian inner product on $\mathbb{C}^n$. In the usual coordinates $(z^k = x^k + iy^k)_{1\leqslant k\leqslant n}$, these are written $h = \sum_{k=1}^n dz_k \otimes d\overline{z'}_k$ and $\omega = \sum_{k=1}^n dx^k \wedge dy^k$. Let $G = \mathrm{U}(1)$ act on M by complex scalar multiplication. The action is Hamiltonian with equivariant moment map $\mu : \mathbb{C}^n \to \mathfrak{u}(1)^* \approx i\mathbb{R}$ given by $\mu(z) = \frac{i}{2}\|z\|^2$. Any value $\alpha = i\frac{r}{2} \in \mathfrak{u}(1)^*$ with $r > 0$ is a regular value of the moment map, and a fixed point of the coadjoint action since G is abelian. Moreover G acts freely and properly (by

is a smooth submanifold of M, and moreover $T_x M_\alpha = \ker d\mu_{|x}$ for any $x \in M_\alpha$. (ii'): M_α/G_α^0 can be given a smooth structure making the projection $p : M_\alpha \to M_\alpha/G_\alpha^0$ a smooth map. Note that when such a smooth structure exits, it is unique.

[72]The reason for the quotation marks around "symplectomorphim" is that the induced 2-form $\iota^*\omega$ on M_α (where $\iota : M_\alpha \to M$ denotes the inclusion) is not a symplectic structure on M_α *a priori*, it can be degenerate. Anyway, "symplectomorphim" means that $p^*\omega_\alpha = \iota^*\omega$.

[73]The more general shift trick is as follows. Let $G \cdot \alpha$ denote the coadjoint orbit of α. It has a natural symplectic structure and a G-equivariant moment map is given by the inclusion $G \cdot \alpha \subset \mathfrak{g}^*$, see example B.2 (3). Consider $\tilde{M} = M \times (G \cdot \alpha)$ with the product symplectic structure and diagonal action of G. Then $\tilde{\mu} : (x, \beta) \mapsto \mu(x) - \beta$ is an equivariant moment map on $\tilde{M}$. Moreover, there is a canonical symplectomorphism between the symplectic reductions $\tilde{\mu}^{-1}(0)/G$ and $\mu^{-1}(\alpha)/G_\alpha$.

compactness) on the level set $\mu^{-1}(\alpha)$, which is the Euclidean sphere S_r of radius r in $\mathbb{C}^n$. For $r = 1$, the symplectic quotient $\mu^{-1}(\alpha)/G$ is the complex projective space $\mathbb{C}\mathbf{P}^{n-1}$ and one can check that the reduced symplectic structure coincides the Kähler form of the Fubini-Study metric on $\mathbb{C}\mathbf{P}^n$. This is no coincidence, as we will see.

B.4 Kähler reduction

Assume now that the connected manifold M has a Kähler structure and that the Hamiltonian action of the connected Lie group G preserves the Kähler structure. Under the restriction that only values $\alpha \in \mathfrak{g}^*$ that are fixed points of the coadjoint action are now allowed, the symplectic quotients $\mu^{-1}(\alpha)/G$ inherit the Kähler structure. Let us now write this down more precisely.

The theorem for Kähler reduction goes:

Theorem B.6. *Let (M, g, I, ω) be a Kähler manifold and let G be a connected Lie group acting on M preserving the Kähler structure*[74]*. Assume that the action is Hamiltonian (with respect to the Kähler form ω) and let $\mu : M \to \mathfrak{g}^*$ be an equivariant moment map. Let $\alpha \in \mathfrak{g}^*$ be a fixed point for the coadjoint action of G such that*

(i) α is a regular value of μ, so that $M_\alpha = \mu^{-1}(\alpha)$ is a G-invariant submanifold of M, and

(ii) the action of G on M_α is free and proper, so that $p : M \to M_\alpha/G$ is a smooth submersion.

Then there is a unique Kähler structure $(g_\alpha, I_\alpha, \omega_\alpha)$ on M_α/G such that $p^* g_\alpha = \iota^* g$ and $p^*\omega_\alpha = \iota^*\omega$[75]*.*

Example B.7. Here are a couple examples of Kähler reduction:

(1) **Fubini-Study metric on $\mathbb{C}\mathbf{P}^n$.** Example B.5 (2) is an example of Kähler reduction, since the action of U(1) on $\mathbb{C}^n$ preserves the Kähler structure. The reduced Kähler structure on $\mathbb{C}\mathbf{P}^n$ can be taken as a definition of the Fubini-Study metric. It is characterized by the fact that the complex structure agrees with the standard complex structure on $\mathbb{C}\mathbf{P}^n$ and the projection map $S_1 \to \mathbb{C}\mathbf{P}^n$ is a Riemannian submersion.

(2) **Complex Grassmannians.** Let $M = \mathbb{C}^{k\times n}$ and let $G = \mathrm{U}(k)$ act on M by matrix multiplication on the left. This action preserves the standard Kähler structure of M and is Hamiltonian. An equivariant moment map $\mu : M \to \mathfrak{g}^*$ is given by $\mu(A) = \frac{i}{2}AA^* \in \mathfrak{u}(k)$ under the identification $\mathfrak{u}(k) \approx \mathfrak{u}(k)^*$ given by the complex Killing form $(a, b) \mapsto -\mathrm{tr}(ab)$. The matrix $\alpha = \frac{i}{2}\mathbf{1}$ is a regular value of μ and a fixed point of the coadjoint action. G acts freely and properly on the level set $\mu^{-1}(\alpha)$, which can be described as the space of unitary k-frames in $\mathbb{C}^n$ (taking the rows of A). It follows that the quotient $\mu^{-1}(\alpha)/G$ is the space of k-planes in $\mathbb{C}^n$, in other words the complex Grassmannian $G(k, n)$, which inherits a Kähler structure by reduction.

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[74]Meaning that the action of G leaves g, I and ω invariant (in other words it is an isometric, holomorphic and symplectic action). By the previous remark, it suffices to say that G preserves two out of these three structures.

[75]Again one is tempted to just say that p is a morphism of Kähler manifolds, but not only $\iota^*\omega$ is very possibly degenerate (so it cannot be called a symplectic structure), M_α is not necessarily a complex submanifold of (M, I), in which case $\iota^* I$ does not make sense. Only $\iota^* g$ is an honest Riemannian metric, so p can at least be called a Riemannian submersion.

B.5 Complex quotients and GIT quotients

Complex quotients

Keep the setting of Kähler reduction: consider a Kähler Hamiltonian action of a connected Lie group K (soon compact) on a Kähler manifold (M, g, I, ω). Also assume that $0 \in \mathfrak{g}^*$ is a regular value of an equivariant moment map $\mu : M \to \mathfrak{g}^*$.

Since the action of K on (M, I) is holomorphic, it is natural to speculate that it uniquely extends to a holomorphic action of the complexified group $K^{\mathbb{C}}$, with infinitesimal action $V : \mathfrak{k}^{\mathbb{C}} = \mathfrak{k} \oplus i\mathfrak{k} \mapsto \Gamma(TM)$ given by $V_{\xi+i\eta} = V_\xi + IV_\eta$. This is not always true though (the flow of these vector fields needs to be complete), but it is when K is compact. From now on we will assume that is the case. One expects that the quotient $\mu^{-1}(0)/K$ may be identified with the quotient $M^{\mathrm{ps}}/K^{\mathbb{C}}$, where $M^{\mathrm{ps}} \subset M$ is the set of points in M whose $K^{\mathbb{C}}$-orbit intersects $\mu^{-1}(0)$. Such points are called *polystable*. This observation gives an alternate viewpoint on Kähler reduction, now described as the quotient of a (generally large) subset of M by the complexified group $K^{\mathbb{C}}$. A finer analysis of the action of $K^{\mathbb{C}}$ on M motivates the following definitions:

Definition B.8. Let M be a connected Kähler manifold with a Kähler Hamiltonian action of a compact connected Lie group K and let $\mu : M \to \mathfrak{g}^*$ be an equivariant moment map for the action. In the sense of symplectic reduction, a point $x \in M$ is called:

- *semistable* if the closure of the $K^{\mathbb{C}}$-orbit of x meets $\mu^{-1}(0)$.
- *polystable* if the $K^{\mathbb{C}}$-orbit of x meets $\mu^{-1}(0)$.
- *stable* if it is polystable and has finite stabilizer under the action of K (equivalently $K^{\mathbb{C}}$).
- *(strictly) unstable* if it is not semistable.

Note that the closure of the orbit of any semistable point contains a unique polystable orbit. Also, note that a point x in $\mu^{-1}(0)$ is stable if and only if it is a regular point for μ, by remark B.4.

Theorem B.9. *Let $M^{\mathrm{s}} \subset M^{\mathrm{ps}} \subset M^{\mathrm{ss}} \subset M$ denote the sets of stable, polystable, and semistable points in M.*

(i) Both M^{s} and M^{ss} are open in M.

(ii) $K^{\mathbb{C}}$ acts properly on M^s and

$$\mu^{-1}(0)_{\mathrm{reg}} \big/ K = M^{\mathrm{s}} \big/ K^{\mathbb{C}} . \tag{B.3}$$

(iii) Denoting $\sim$ the equivalence relation on M^{ss} identifying two points when the closures of their $K^{\mathbb{C}}$-orbits intersect,

$$\mu^{-1}(0) \big/ K = M^{\mathrm{ss}} \big/ \sim = M^{\mathrm{ps}} \big/ K^{\mathbb{C}} . \tag{B.4}$$

In particular, when 0 is a regular value of μ and K acts freely on $\mu^{-1}(0)$, then $M^{\mathrm{ps}} = M^{\mathrm{s}}$ and

$$M^{\mathrm{ss}} \big/ \sim = M^{\mathrm{s}} \big/ K^{\mathbb{C}} = M /\!\!/ K \tag{B.5}$$

where $M /\!\!/ K$ denotes the symplectic quotient.

We will see examples of complex quotients in what follows, *e.g.* example B.19.

GIT quotients

In the algebraic setting, this relates to *Geometric Invariant Theory* (in short *GIT*). A decent introduction to GIT is well beyond the scope of these notes, but let us try to give a sense of what it is and how it relates to symplectic reduction and complex quotients. The standard reference for GIT is [36], a nice reference for our purpose is [48].

So, assume now that M has the structure of an algebraic variety over the complex numbers and G is a complex algebraic group acting on M. The purpose of GIT is to define a quotient of M by the action of G with an algebro-geometric structure. GIT shows that when G is reductive (*i.e.* $G = K^{\mathbb{C}}$ with K compact connected), one can construct an open subvariety $M^{\mathrm{ss}} \subset M$ and a *categorical quotient*[76] for the G-action on M^{ss}.

Let us first look at the easy case where M is an affine variety. Let $\mathbb{C}[M]$ denote the coordinate ring of regular functions on M and $\mathbb{C}[M]^G \subset \mathbb{C}[M]$ the subalgebra of G-invariant functions. $\mathbb{C}[M]^G$ is finitely generated provided G is reductive by Nagata's theorem solving Hilbert's 14th problem. Therefore $\mathbb{C}[M]^G$ is the coordinate ring of an affine variety (namely $\operatorname{Spec} \mathbb{C}[M]^G$). This variety is called the *(affine) GIT quotient* of M and denoted $M/\!/G$.

Now, note that the inclusion $\mathbb{C}[M]^G \to \mathbb{C}[M]$ defines a morphism of affine varieties $\pi\colon M \to M/\!/G$. One can show that it is a *good quotient*[77], which implies that (1) π is surjective, (2) two points in M are in the same fiber if and only if the closures of their G-orbits intersect, and (3) any fiber of π contains a unique closed orbit. In view of that, we give the following definitions and theorem:

Definition B.10. In the sense of GIT, a point $x \in M$ is called:
- *polystable* if its orbit $G \cdot x$ is closed.
- *stable* if x is polystable and has finite stabilizer.

Theorem B.11. *Let $M^{\mathrm{s}} \subset M^{\mathrm{ps}} \subset M$ denote the sets of stable and polystable in M. M^{s} is open in M. Let $\sim$ denote the equivalence relation on M identifying two points when the closures of their G-orbits intersect. Then (at least as topological spaces)*

$$M/\!/G = M/_{\sim} = M^{\mathrm{ps}}/G \tag{B.6}$$

and M^s/G is an open set in $M/\!/G$[78].

Example B.12. Here are a couple examples of affine GIT quotients:

(1) Let $G = \mathbb{C}^*$ act on $M = \mathbb{C}^n$ by scalar multiplication. The only invariant functions are the constants, so $M/\!/G = \{0\}$ is a point (and not $\mathbb{C}\mathbf{P}^{n-1}$ as one could hope). This corresponds to 0 being the only polystable point in M. This example shows that the affine case is sometimes best embedded in the projective case in a nontrivial way.

(2) Let $G = \mathrm{GL}_n(\mathbb{C})$ act on $M = \mathcal{M}_{n\times n}(\mathbb{C})$ by conjugation. The coefficients of the characteristic polynomial $P(\lambda) = \det(\lambda\mathbf{1} - (X_{ij}))$ are invariant functions on M, in fact they generate $\mathbb{C}[M]^G$ (hint: diagonalizable matrices are Zariski dense). Therefore the affine GIT quotient $M/\!/G$ is $\mathbb{C}^n$, with projection $\pi : M \to \mathbb{C}^n$ given by the coefficients of the characteristic polynomial of a matrix. The polystable points of M are the diagonalizable matrices, and there are no stable points.

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[76] I will not give a general definition of categorical quotients, the interested reader may refer to [36].

[77] It is also a *categorical quotient*. See [36] for details.

[78] Moreover the projection $\pi : M^s \to \pi(M^s) \subset M$ is a *geometric quotient*, even better than a good quotient.

(3) **Character varieties.** Let G be a linear complex reductive group. That includes these notes' favorite group $G = \mathrm{PSL}_2(\mathbb{C})$ since $\mathrm{PSL}_2(\mathbb{C}) \approx SO_3(\mathbb{C})$. Let Γ be a finitely generated group and $M = \mathrm{Hom}(\Gamma, G)$. M is an affine variety (called the representation variety) and G acts on M by conjugation. The affine GIT quotient $\mathcal{X}(\Gamma, G) := \mathrm{Hom}(\Gamma, G)//G$ is called the G-character variety of Γ. The polystable points of M are precisely the reductive representations, so as a topological space

$$\mathcal{X}(\Gamma, G) = \mathrm{Hom}(\Gamma, G) /_{\sim} = \mathrm{Hom}^{\mathrm{reductive}}(\Gamma, G) /_{G} . \tag{B.7}$$

The stable points of M are the irreducible representations, thus the conjugacy classes of irreducible representations embed as an open set in $\mathcal{X}(\Gamma, G)$.

Now let us summarily cover the case where M is a projective variety acted on by a reductive algebraic group G. This is the standard setting in GIT. For simplicity assume that an embedding of M in complex projective space $\mathbb{C}\mathbf{P}^n = \mathbf{P}(\mathbb{C}^{n+1})$ has been fixed and that the action of G is projective linear with a fixed *linearization*, meaning that G acts via a given morphism $G \longrightarrow \mathrm{GL}(n+1, \mathbb{C})$. Let $\hat{M} \subset \mathbb{C}^{n+1}$ denote the affine cone over M. The (points of the) projective variety M is identified to the (maximal) projective spectrum $\mathrm{Proj}\,\mathbb{C}[\hat{M}]$[79]. The action of G acts on the graded algebra $\mathbb{C}[\hat{M}]$ preserves the grading, so the ring of invariant functions $\mathbb{C}[\hat{M}]^G$ is a reduced graded subalgebra. By Nagata's theorem this is also finitely generated, so there is a projective variety with coordinate ring $\mathbb{C}[\hat{M}]^G$ (namely $\mathrm{Proj}\,\mathbb{C}[\hat{M}]^G$). This variety is called the *(projective) GIT quotient* of M, denoted $M//G$.

The inclusion $\mathbb{C}[\hat{M}]^G \longrightarrow \mathbb{C}[\hat{M}]$ induces a rational map $\pi : M \dashrightarrow M//G$, which is undefined on the *nilcone* $M^{\mathrm{u}} \subset M$ comprising the points $x \in M$ such that $f(\hat{x}) = 0$ for all non-constant homogeneous forms $f \in \mathbb{C}[\hat{M}]^G$ (where $\hat{x} \in \mathbb{C}^{n+1} - \{0\}$ is a point lying over x). In other words the points of M^{u} are the points of M which are not "seen" in the GIT quotient $M//G$. Throwing away these points, the restriction $\pi \colon M - M^{\mathrm{u}} \longrightarrow M//G$ is a well-defined morphism of projective varieties, and one can show that it is a *good quotient*. Let us record the following definitions and theorem:

Definition B.13. In the sense of GIT, a point $x \in M$ is called:

- *semistable* if $x \in M - M^{\mathrm{u}}$. Equivalently, for a (hence all) $\hat{x} \in \mathbb{C}^{n+1} - \{0\}$ lying over x, the orbit closure $\overline{G \cdot \hat{x}}$ does not contain the origin $0 \in \mathbb{C}^{n+1}$.
- *polystable* if x is semistable and its orbit $G \cdot x$ is closed in the semistable locus. Equivalently, the orbit $G \cdot \hat{x}$ is closed in for a (hence all) $\hat{x} \in \mathbb{C}^{n+1} - \{0\}$ lying over x.
- *stable* if x is polystable and has finite stabilizer.
- *(strictly) unstable* if $x \in M^{\mathrm{u}}$.

Theorem B.14. *Let $M^{\mathrm{s}} \subset M^{\mathrm{ps}} \subset M^{\mathrm{ss}} \subset M$ denote the sets of stable, polystable, and semistable points in M. Both M^{s} and M^{ss} are open in M. Let $\sim$ denote the equivalence relation on M^{ss} identifying two points when the closures of their G-orbits intersect. Then*

$$M /\!/_{G} = M^{\mathrm{ss}} /_{\sim} = M^{\mathrm{ps}} /_{G} \tag{B.8}$$

and M^s/G is an open set in $M//G$.

[79] I will not recall what this means, the uninformed reader may ignore this part or look up [23].

Example B.15. Let us look again at the case where $G = \mathbb{C}^*$ acts on $M = \mathbb{C}^n$ by scalar multiplication. Let us embed M in $\mathbb{C}\mathbf{P}^n$ classically by $(z_1, \dots, z_n) \mapsto [z_1 : \cdots : z_n : 1]$. If we consider the trivial linearization of the action given by $G \to \mathrm{GL}(n+1, \mathbb{C})$, $\lambda \mapsto \mathrm{diag}(\lambda, \dots, \lambda, 1)$, then we recover the affine case B.12 (1) where the GIT quotient is a single point. Instead, let us consider different linearizations associated to characters $\chi_p : \mathbb{C}^* \to \mathbb{C}^*, \lambda \mapsto \lambda^{-p}$ by $G \to \mathrm{GL}(n+1, \mathbb{C})$, $\lambda \mapsto \mathrm{diag}(\lambda, \dots, \lambda, \chi_p(\lambda))$.

- For $p = 0$, this is just the trivial linearization discussed above.
- For $p < 0$, there are no non-constant invariant functions in $\mathbb{C}[\hat{M}]$ so $M/\!/G$ is empty. Accordingly, all points are unstable.
- For $p > 0$, the invariant homogeneous functions on $\hat{M}$ are of the form $f(z_1, \dots, z_{n+1}) = g(z_1, \dots, z_n) z_{n+1}^{mp}$, where $m \geqslant 0$ and g is a homogeneous polynomial of degree mp in n variables. Check that $M/\!/G = \mathrm{Proj}\, \mathbb{C}[\hat{M}] = \mathbb{C}\mathbf{P}^{n-1}$. Accordingly, $0 \in M$ is the only unstable point and all other points are polystable (in fact stable), so that $M/\!/G = (\mathbb{C}^n - \{0\})/\mathbb{C}^* = \mathbb{C}\mathbf{P}^{n-1}$.

Kähler reduction and GIT quotients

Finally let us relate Kähler reduction to GIT quotients. So, let M be a connected Kähler manifold with a Kähler Hamiltonian action of a connected compact Lie group K. Assume that M can be embedded as a complex projective subvariety of $\mathbb{C}\mathbf{P}^n$ for some $n > 0$ in a such a way that:

(1) The Kähler structure on M is the restriction of the Fubini-Study metric to M.
(2) K acts on M via a morphism $\rho : K \to \mathrm{U}(n+1)$.

We recall the Fubini-Study metric is the standard Kähler structure of $\mathbb{C}\mathbf{P}^n$ (we saw in example B.7 (1) that it may be described as a Kähler reduction of the standard Hermitian metric on $\mathbb{C}^{n+1}$) and that it is preserved by the action of $\mathrm{U}(n+1)$. Any complex subvariety of $\mathbb{C}\mathbf{P}^n$ inherits a Kähler structure by restriction. An equivariant moment map μ for the action is determined up to an element of $\alpha \in \mathrm{H}^1(\mathfrak{k}, \mathbb{R})$ in general, but here there is a natural choice that is taking the moment map induced by the "standard" moment map for the action of $\mathrm{U}(n+1)$ on $\mathbb{C}\mathbf{P}^n$[80].

Remark B.16. Out of interest, we can write down some explicit formulas. Let $[z_1 : \cdots : z_{n+1}]$ denote homogeneous coordinates on $\mathbb{C}\mathbf{P}^n$ and $z = (z_1, \dots, z_{n+1})$. The Fubini-Study Hermitian metric is given by

$$h_{\mathrm{FS}} = \frac{1}{\|z\|^4} \left(\|z\|^2 \sum_{i=1}^{n+1} dz_i \otimes d\overline{z_i} - \sum_{1 \leqslant j,k \leqslant n+1} \overline{z_j} z_k \, dz_j \otimes d\overline{z_k} \right) \tag{B.9}$$

and the Fubini-Study Kähler form is

$$\omega_{\mathrm{FS}} = -\,\mathrm{Im}(h_{\mathrm{FS}}) = \frac{i}{2} \partial \overline{\partial} \log \|z\|^2 . \tag{B.10}$$

[80] It is worth mentioning that choosing a different embedding and linearization of the action amounts to choosing different values for the moment map.

The standard moment map for the action of $\mathrm{U}(n+1)$ on $\mathbb{C}\mathbf{P}^n$ is the map $\tilde{\mu}\colon \mathbb{C}\mathbf{P}^n \to \mathfrak{u}(n+1)^*$ given by

$$\langle \tilde{\mu}(z), \xi \rangle = \frac{i}{2} \frac{\operatorname{tr}(z^* \xi z)}{\|z\|^2} \tag{B.11}$$

where z^* denotes the conjugate transpose of the column vector $z \in \mathbb{C}^{n+1}$. Note that under the identification $\mathfrak{u}(n+1) \approx \mathfrak{u}(n+1)^*$ given by the pairing $(\xi, \xi') \mapsto -\operatorname{tr}(\xi\xi')$, this is

$$\tilde{\mu}(z) = \frac{1}{2i} \frac{zz^*}{\|z\|^2}\,. \tag{B.12}$$

The induced moment map for the action of K on M is the map μ as in the following diagram:

$$\begin{array}{ccc} \mathbb{C}\mathbf{P}^n & \xrightarrow{\ \tilde{\mu}\ } & \mathfrak{u}(n+1)^* \\ \uparrow & & \downarrow{\scriptstyle r^*} \\ M & \xrightarrow{\ \mu\ } & \mathfrak{k}^* \end{array} \tag{B.13}$$

where $r : \mathfrak{k} \to \mathfrak{u}(n+1)$ denotes the derivative of $\rho : K \to \mathrm{U}(n+1)$.

Now, recall that the action of K extends to a holomorphic action of $K^{\mathbb{C}}$, in this case via the unique $\rho^{\mathbb{C}}\colon K^{\mathbb{C}} \to \mathrm{U}(n+1)^{\mathbb{C}} = \mathrm{GL}(n+1,\mathbb{C})$. We saw how the symplectic notion of stability (definition B.8) is the bridge between Kähler reduction and complex quotients (theorem B.9). We also saw how the GIT notion of stability (definition B.13) relates GIT quotients to complex quotients in the algebraic setting (theorem B.14). The picture is completed by the Kempf-Ness theorem:

Theorem B.17. *In the setting above, the definitions of semistable, polystable, stable and unstable points in M in the sense of symplectic reduction (definition B.8) coincide with the homonymous notions in the sense of GIT (definition B.13).*

Identifications of quotients follows from theorem B.9 and theorem B.14. In particular:

Corollary B.18. *If 0 is a regular value for the moment map and K acts freely on $\mu^{-1}(0)$, then*

$$M /\!/ K = M^{\mathrm{s}} / K^{\mathbb{C}} = M /\!/ K^{\mathbb{C}} \tag{B.14}$$

where $M/\!/K$ denotes the symplectic quotient and $M/\!/K^{\mathbb{C}}$ the GIT quotient.

Let us look at this circle of ideas at work on a simple example:

Example B.19. We look at example B.15: $M \approx \mathbb{C}^n$ is the projective variety $\{[z_1 : \cdots : z_n : 1]\} \subset \mathbb{C}\mathbf{P}^n$ and $G = \mathbb{C}^*$ acts on $\mathbb{C}\mathbf{P}^n$ preserving M via the representation $\rho : G \to \mathrm{GL}(n+1,\mathbb{C})$, $\lambda \mapsto \operatorname{diag}(\lambda, \ldots, \lambda, \lambda^{-p})$. We saw from the GIT point of view that

- when $p < 0$ all points are unstable and the GIT quotient $M/\!/G$ is empty;
- when $p = 0$ the only (poly)stable point is $x_0 = [0 : \cdots : 0 : 1]$ and $M/\!/G$ is reduced to a point;
- when $p > 0$ all points are (poly)stable except x_0 and $M/\!/G = \mathbb{C}\mathbf{P}^{n-1} \approx (\mathbb{C}^n - \{0\})/\mathbb{C}^*$.

From the perspective of symplectic (Kähler) reduction, $M \approx \mathbb{C}^n$ with Kähler form given by the restriction of ω_{FS}[81] and the action of the maximal compact $K = U(1)$ is the action by scalar multiplication, which is Kähler Hamiltonian. We compute the equivariant moment map μ using (B.11) and (B.13). The derivative of ρ at the identity is $r : \mathfrak{k} \approx i\mathbb{R} \to \mathfrak{u}(n+1)$, $ix \mapsto \operatorname{diag}(ix, \ldots, ix, -ipx)$. It follows that μ is given by

$$\begin{aligned} \mu \colon M \approx \mathbb{C}^n &\longrightarrow \mathfrak{u}(1)^* \approx i\mathbb{R} \\ z = (z_1, \ldots, z_n) &\longmapsto \frac{1}{2i} \frac{\|z\|^2 - p}{1 + \|z\|^2}. \end{aligned}$$

- For $p < 0$, $\mu^{-1}(0)$ is empty so the symplectic quotient wannabe $\mu^{-1}(0)/U(1)$ is empty.
- For $p = 0$, $\mu^{-1}(0) = \{0\}$ so $\mu^{-1}(0)/U(1)$ is a point.
- For $p > 0$, 0 is a regular value of μ, $\mu^{-1}(0)$ is the sphere $S_{\sqrt{p}}$ of radius $\sqrt{p}$ in $\mathbb{C}^n$ and $U(1)$ acts freely on it so the well-defined symplectic quotient is $M/\!/K = S_{\sqrt{p}}/U(1)$. In accordance with theorem B.9, we find that all points in M except the origin are stable in the sense that their $\mathbb{C}^*$-orbit intersects $S_{\sqrt{p}}$, and the action of $\mathbb{C}^*$ on $M^{\mathrm{s}} = (\mathbb{C}^n - \{0\})$ is free and proper so that

$$M/\!/K = S_{\sqrt{p}} \Big/ U(1) \approx \mathbb{C}^n - \{0\} \big/ \mathbb{C}^* \approx \mathbb{C}\mathbf{P}^{n-1}. \tag{B.15}$$

  Since this notion of stability coincides with that of GIT, the complex quotient $\mathbb{C}^n - \{0\}/\mathbb{C}^*$ is also identified to the GIT quotient $M/\!/G = \mathbb{C}\mathbf{P}^{n-1}$ as expected.

B.6 Complex symplectic reduction

In preparation.

B.7 Hyperkähler reduction

In preparation.

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